Sequence of Syracuse : Extensions of types $3n + b$ and $5n + 1$

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Abstract

We investigate two families of extensions of the well-known Syracuse sequence. In the standard case, starting from an initial integer $v_0 > 0$, the sequence (v_n) is defined by the recurrence: $v_{n+1} = (3v_n + 1)/2$ if v_n is odd, and $v_n/2$ if v_n is even.

For the generalizations studied here, we denote the sequence by (V_n) to clearly distinguish it from the classical case. In all cases, the rule applied to even integers remains $V_{n+1} = V_n/2$. We consider two types of extensions:

(i) $3n + b$ *extensions*: for a fixed odd integer b, the transformation becomes $V_{n+1} = (3V_n + b)/2$ when V_n is odd;

(ii) *The* $5n + 1$ *extension*: the rule becomes $V_{n+1} = (5V_n + 1)/2$ when V_n is odd.

Building upon results established in the classical case (when $b = 1$), we show that:

- for every odd integer b, the $3n + b$ extension admits only finitely many cycles, all of which can be detected by testing initial values $V_0 \leq 2^{f(b)} \cdot |b|$. Moreover, no sequence diverges. In particular, when $|b| < 2^{10}$, the bound $f(b) = 48$ suffices;
- in contrast, the $5n + 1$ extension admits at least one divergent orbit.

1 Introduction

The Syracuse (or Collatz) sequence is defined on $\mathbb{N}$ by the following rule, applied to an initial integer $v_0 > 0$:

$$v_{n+1} = \begin{cases} v_n/2, & \text{if } v_n \text{ is even,} \\ (3v_n + 1)/2, & \text{if } v_n \text{ is odd.} \end{cases}$$

The Syracuse conjecture asserts that every such sequence eventually reaches the value 1 in a finite number of steps. A complete proof of this result was proposed by the author in a previous work [1].

In this paper, we extend the underlying dynamical process to two families of affine transformations. To clearly distinguish the standard case from the generalized setting, we adopt the following convention:

- The lowercase letter v denotes the sequence in the classical case;
- The uppercase letter V is used for sequences arising from the studied extensions.

In all cases, the rule applied to even integers remains unchanged: $V_{n+1} = V_n/2$ when V_n is even.

We consider the following two types of extensions:

- $3n + b$ **extensions**: for a fixed odd integer b, the transformation applied to odd integers becomes

$$V_{n+1} = (3V_n + b)/2, \quad \text{if } V_n \text{ is odd.}$$

- **The** $5n + 1$ **extension**: in this case, the transformation becomes

$$V_{n+1} = (5V_n + 1)/2, \quad \text{if } V_n \text{ is odd.}$$

For the $3n+b$ extensions, the structure of the proof closely parallels that of the standard case ($b = 1$) [1]. In particular, the tools developed in the proof of the Syracuse conjecture—ranking functions, logarithmic bounds, and the *Random List Theorem*—are almost entirely reusable. We prove that all such sequences are convergent and that the complete set of cycles can be explored within a bounded domain that depends on b.

It should be noted, however, that when $b < 0$, the proof becomes significantly more involved. Several properties used in the $b > 0$ case cannot be directly applied, and key arguments must be reconstructed in a symmetric framework.

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The $5n+1$ extension, in contrast, exhibits divergent orbits. Once again, we make use of the *Random List Theorem* established in the classical case, but a careful adaptation of the enumeration of candidate lists is required—this being the most delicate part of the new proof.

Once these limiting cases are addressed, it is natural to consider more general sequences defined by

$$V_{n+1} = \begin{cases} \dfrac{V_n}{2}, & \text{if } V_n \text{ is even,} \\ \dfrac{aV_n + b}{2}, & \text{if } V_n \text{ is odd, where } a \text{ and } b \text{ are odd.} \end{cases}$$

or, more generally, by allowing both parameters a and b to be even:

$$V_{n+1} = \begin{cases} \dfrac{V_n}{2}, & \text{if } V_n \text{ is even,} \\ \dfrac{aV_n + b}{2}, & \text{if } V_n \text{ is odd, where } a \text{ and } b \text{ are even.} \end{cases}$$

The asymptotic behavior of these general extensions can then be analyzed using similar principles, with suitable adjustments involving the logarithmic constant $X = \dfrac{\ln 2}{\ln a - \ln 2}$, the presence of trivial cycles for certain values of b, and the divergent or non-divergent nature of the trajectories depending on the value of a.

2 The $3n+b$ Extensions

2.1 Definitions

2.1.1 Syracuse Sequence with Parameter b: V

Let b be a fixed odd integer. We define the sequence (V_n) by the recurrence:

$$V_{n+1} = \begin{cases} V_n/2, & \text{if } V_n \text{ is even} \quad (\text{type 0 transition}), \\ (3V_n + b)/2, & \text{if } V_n \text{ is odd} \quad (\text{type 1 transition}). \end{cases}$$

For the sake of precision, this sequence can also be defined via iteration of the function T_b, that is, $V_n = T_b^{(n)}(V_0)$.

Remarks.

- An alternative approach is to restrict to $b > 0$ and allow $V_0 < 0$. Indeed, when $b < 0$, if we define $\overline{V}_0 = -V_0 < 0$ and $\overline{b} = -b > 0$, then by a straightforward induction one obtains $\overline{V}_n = -V_n$ for all n.
- This representation will be used later to simplify the proof in the case $b < 0$.
- To simplify notation, the dependence of V on b will not always be made explicit.

2.1.2 Approximated Sequence: V'

We define the approximated sequence (V'_n) by:

$$V'_{n+1} = \begin{cases} V'_n/2, & \text{if } V_n \text{ is even,} \\ (3V'_n)/2, & \text{if } V_n \text{ is odd.} \end{cases}$$

We introduce a remainder term R_n defined by $V_n = V'_n + R_n$, whose detailed analysis will be carried out later.

2.2 Main Theorem for the $3n+b$ Extensions

Théorème 2.1 (Extensions $3n+b$, odd $b > 0$)**.** *There exists an integer $N_0 = N_0(b)$ such that, for any initial value $V_0 \le 2^{N_0}|b|$, the sequence (V_n) eventually enters a finite cycle. In particular, no such sequence diverges for any $V_0 > 0$.*

When $|b| < 2^{10}$, a preliminary verification up to $N_0 = 48$ suffices to rule out the existence of nontrivial cycles. As $|b|$ increases, the required bound N_0 to ensure this exclusion grows slowly. Beyond a certain threshold, even this initial verification becomes impractical.

2.2.1 Proof in the Case $0 < b < 1024$

Proof. We follow the complete proof developed for the standard case ($b = 1$) [1], indicating the specific modifications required for the $(3n + b)$ extension.

- **Lemma 3.1.** The probability that V_1 is even remains 1/2 for $V_0 > 4$. One only needs to include the binary decomposition of b in the probabilistic analysis.
- **Proposition 4.1.** We immediately have $R_n = b \cdot r_n$, which alters all subsequent upper bound formulas involving R_n.
- **Lists** $\mathrm{JGL}(N, V_0)$**.**
  - The definition remains unchanged.
  - A very trivial cycle of length 2 is observed for $V_0 = b$, corresponding to the transition list 10: we have $V_1 = (3b + b)/2 = 2b$ and then $V_2 = b$.
  - The values of n remain unchanged; the corresponding $\mathrm{VMax}(n)$ values are simply multiplied by $|b|$.
  - The record values of $\mathrm{VMax}(n)$ are reached at the same indices as in the case $b = 1$.
  - The constant c_α becomes c'_α in the case $b > 0$, since the maximum values $\mathrm{VMax}(n)$ are scaled by $|b|$.

    In the standard case, it is shown that for $\alpha > 48$, a constant $c_\alpha = \lceil 2{,}017{,}000\,\alpha \rceil$ is sufficient to bound the size of the random transition lists required to guarantee the existence of at least $p + 1$ solutions.

    In the general case $b > 0$, the growth of $\mathrm{VMax}(n)$ requires considering an adjusted index $\alpha' = \alpha + \log_2 b$, since a multiplicative factor b on the integers translates into an additive shift in base-2 logarithmic scale.

    Hence, we obtain:
    $$c'_\alpha = 2{,}017{,}000\alpha > 2{,}017{,}000(\alpha' - \log_2 b).$$
    Since we assume $b < 2^{10}$, it follows that $\log_2 b < 10$, so
    $$c'_\alpha > 2{,}017{,}000(\alpha' - 10).$$
    Moreover, given that $\alpha' > 48$, we deduce:
    $$c'_\alpha > 2{,}017{,}000\left(\alpha' - \frac{10\alpha'}{48}\right) = 1{,}596{,}791\alpha'.$$
    Likewise, given that $\alpha' > 22$, it follows that:
    $$c'_\alpha > 639\left(\alpha' - \frac{10\alpha'}{22}\right) > 347\,\alpha'.$$
    This explicit bound allows us to preserve all the asymptotic properties used in the proof, especially in the analysis of the parameter e in the Random List Theorem.
- **Testing the conjecture up to** $V_0 \leq 2^\alpha |b|$**.** For $0 < b < 1024$ and $\alpha = 22$ (i.e., $V_0 < 2^{22} b$), the following results are obtained:
  - 259 values of b admit only the very trivial cycle;
  - the longest cycle has length 426 and occurs for $b = 563$, $V_0 = 19$;
  - 2864 trivial cycles were identified.

  The complete tests and results are available online [2].

  Remarque 2.2. If we apply the Central Limit Theorem version of the *Random List Theorem*, choosing $\alpha = 22$ instead of 48 is sufficient. The tests become significantly shorter while leading to essentially the same conclusion.
- **Elimination of certain transition lists.** Some transition lists correspond to orbits starting at a value $W_0 < v_0$ that eventually reach a cycle whose minimal value is v_0. In such cases, the associated transition list must be discarded.

  For instance, for $b = 5$, the cycle {19, 31, 49, 76, 38} (corresponding to the list 11100) is reached by the orbit starting at $W_0 = 3$. Thus, for a list of length $N = 22$, one must exclude the transition list

  $$11101\,11100\,11100\,11100\,11$$

  in which we identify:
  - the initial transitions 11101 allowing entry into the cycle from $W_0 = 3$;
  - multiple repetitions of the cycle pattern 11100;
  - a final truncated pattern due to the specific value of N.

This phenomenon concerns at most a few transition lists for each value of b within the considered interval. The complete list of such cases can be found in the online numerical results [2].

- **Inductive step.** We apply the Random List Theorem with $\alpha > 48$ and $p = 1{,}700{,}000$. Let $n = \alpha + p$, $N = \lceil 1{,}596{,}791\,\alpha \rceil$, and define

$$f(N) < 1.2\,p + 0.95\,N.$$

Then we compute:

$$e = n - N + f(N),$$
$$e < n + 1.2\,p - 0.05\,N = 2.2\,p - 79{,}838\,\alpha.$$

For $\alpha \geq 48$, this yields $e < -92{,}224 \ll 20$. According to the theorem, this implies that at most 1,052,383 solutions can be found with the probability of $1 - 10^{-3}$ using the Berry–Esseen inequality, while we expect at least 1,700,000. This contradiction validates the recurrence step.

Remarque 2.3. If we apply the *Random List Theorem* with $\alpha > 22$ and $p = 200$, and let $n = \alpha + p$ and $N = \lceil 437\,\alpha \rceil$, then we obtain: $e < -39 \ll 6$. Using the CLT part of the theorem, this leads to a contradiction since $201 \gg 96$.

□

2.2.2 Proof in the Case $-1024 < b < 0$

Proof. As in the standard case ($b = 1$) [1], the proof relies on the same general structure, but requires significant adjustments due to the asymmetry introduced by a strictly negative parameter b.

- **Reduction to the case $b > 0$.** We consider the associated sequence $\overline{V}_n = -V_n$, where $\overline{b} = -b > 0$ and $\overline{V}_0 = -V_0 < 0$. This yields a symmetric dynamical system whose behavior is analogous to the case $b > 0$.
- **Transition structure.** The sequence $\overline{V}_n$ satisfies the same recurrence:

$$\overline{V}_{n+1} = \begin{cases} \overline{V}_n/2, & \text{if } \overline{V}_n \text{ is even,} \\ (3\overline{V}_n + \overline{b})/2, & \text{if } \overline{V}_n \text{ is odd.} \end{cases}$$

- **Adapted definition of** $\mathrm{JGL}(N, V_0)$**.** The construction of the threshold $\mathrm{JGL}(N, V_0)$ is based on the symmetrized sequence $\overline{V}_n$, using either the maximum among the first N values of the orbit or, more conveniently, the minimum of their absolute values. This approach exploits the structural symmetry of the dynamical system and allows the reuse of tools developed for the case $b > 0$.

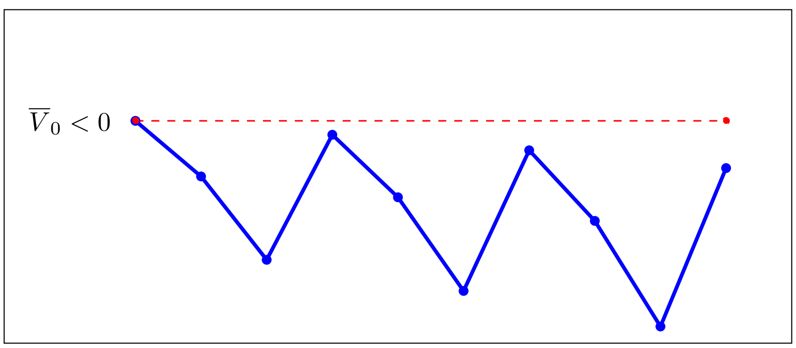

Figure 1: Evolution of the sequence $(\overline{V}_n)$ for a negative initial value $\overline{V}_0 < 0$.

  - **Construction of the list** $\mathrm{JGL}(n+1)$**.** Starting from the expression

$$\overline{V}_n = \frac{3^{m_n}}{2^n} \cdot \overline{V}_0 + \overline{R}_n,$$

we examine whether a type 0 transition is possible at the next step:

$$\overline{V}_{n+1} = \frac{\overline{V}_n}{2} \leq \overline{V}_0,$$

which is equivalent to:

$$\left(1 - \frac{3^{m_n}}{2^{n+1}}\right)\overline{V}_0 \geq \frac{\overline{R}_n}{2}.$$

Two cases arise:

  * If $\frac{3^{m_n}}{2^{n+1}} < 1$, then the inequality is always false (since $\overline{V}_0 < 0$ and $\overline{R}_n > 0$), so $t_n = 1$.

* If $\frac{3^{m_n}}{2^{n+1}} > 1$, then a type 0 transition is possible if

$$\overline{V}_0 \leq \frac{2^n \overline{R}_n}{2^{n+1} - 3^{m_n}} = \mathrm{VMax}(n).$$

  We distinguish the following subcases:
  · If $\mathrm{VMax}(n) > \overline{V}_0$, then $t_n = 0$ is possible: the transition matches that of $\mathrm{Ceil}(n+1)$;
  · If $\mathrm{VMax}(n) = \overline{V}_0$, then we obtain a cycle;
  · If $\mathrm{VMax}(n) < \overline{V}_0$, then $t_n = 1$, and the transition differs from that of $\mathrm{Ceil}(n+1)$.

– **Numerical results for $b = -1$.** A numerical test performed up to $N =$ 500,000 with 20-digit precision identifies the values of n for which $\mathrm{VMax}(n)$ reaches a record. These correspond to potential divergence points compared to the behavior predicted by $\mathrm{Ceil}(n)$. The table below summarizes the key data with $c = n \cdot \ln 2 / \ln(\lfloor \mathrm{VMax}(n) \rfloor)$:

k	m	d	$n = m + d$	$\mathrm{VMax}(n)$	c
1	2	0	2	$5 \approx 2^{2.322}$	8.614e-1
2	7	3	10	$27.10 \approx 2^{4.760}$	2.103
3	12	6	18	$219.31 \approx 2^{7.777}$	2.315
4	53	30	83	$6143.16 \approx 2^{12.585}$	6.595
5	106	61	167	$6143.16 \approx 2^{12.585}$	13.270
6	265	154	419	$6143.16 \approx 2^{12.585}$	33.294
7	318	185	503	$6143.16 \approx 2^{12.585}$	39.969
8	359	209	568	$81063.35 \approx 2^{16.307}$	34.832
9	665	388	1053	$\approx 2^{21.805}$	48.291
10	3990	2333	6323	$\approx 2^{21.805}$	289.976
11	7980	4667	12647	$\approx 2^{21.805}$	579.997
12	10640	6223	16863	$\approx 2^{21.805}$	773.344
13	16266	9514	25780	$\approx 2^{27.195}$	947.974
14	31867	18640	50507	$\approx 2^{29.974}$	1685.015
15	95601	55922	151523	$\approx 2^{29.974}$	5055.112
16	111202	65048	176250	$\approx 2^{32.790}$	5375.095
17	222404	130097	352501	$\approx 2^{32.790}$	10750.221
18	301739	176505	478244	$\approx 2^{34.256}$	13960.753
66	32153285	18808465	50961750	$\approx 2^{47.979}$	1062169.9
67	42934559	25115106	68049665	$\approx 2^{49.227}$	1382371.4

Remarque 2.4 (Structure of the Approximants)**.** The quotients $\frac{m_n}{d_n+1}$, which govern significant changes in the growth of $\mathrm{VMax}(n)$, correspond exactly to the upper approximations of the real number

$$X = \frac{\ln 2}{\ln 3 - \ln 2}.$$

All other fractions that appear throughout the iterations are non-reduced multiples of these approximants and have no influence on the global dynamics.

The purpose of the following paragraphs is to justify this observation through the analysis of the numerical results presented above.

– **Upper Approximations of X via the Stern–Brocot Tree.**
Our aim is to identify the values of n for which $\mathrm{VMax}(n)$ reaches a new record and to establish that these correspond precisely to the upper rational approximations of the real number

$$X = \frac{\ln 2}{\ln 3 - \ln 2}$$

by reduced fractions $\frac{m}{d}$.

To do so, we use the classical method of the Stern–Brocot tree:

* We begin with the initial bounds $\frac{a}{b} = \frac{0}{1}$ and $\frac{c}{d} = \frac{1}{0}$.
* At each step, we consider the mediant $\frac{a+c}{b+d}$ and compare it to X:
  · If $X \leq \frac{a+c}{b+d}$, then this new fraction becomes a better upper approximation of X, and we update the upper bound: $\frac{c}{d} \leftarrow \frac{a+c}{b+d}$.
  · If $X > \frac{a+c}{b+d}$, then we update the lower bound: $\frac{a}{b} \leftarrow \frac{a+c}{b+d}$.
* This process is repeated until the desired precision is reached.

The following numerical results correspond to the first few upper approximations of X:

k	m	d	N	diff $= \frac{m}{d} - X$	$d^2\lvert\text{diff}\rvert$	$d^3\lvert\text{diff}\rvert$
1	2	1	3	-2.9049$\times10^{-1}$	2.9049$\times10^{-1}$	2.90$\times10^{-1}$
2	7	4	11	-4.0489$\times10^{-2}$	6.4782$\times10^{-1}$	2.59
3	12	7	19	-4.7744$\times10^{-3}$	2.3395$\times10^{-1}$	1.64
4	53	31	84	-1.6613$\times10^{-4}$	1.5965$\times10^{-1}$	4.95
5	359	210	569	-1.2518$\times10^{-5}$	5.5205$\times10^{-1}$	115.93
6	665	389	1054	-2.7677$\times10^{-7}$	4.1881$\times10^{-2}$	16.29
7	16266	9515	25781	-6.5991$\times10^{-9}$	5.9746$\times10^{-1}$	5684.79
8	31867	18641	50508	-9.6119$\times10^{-10}$	3.3400$\times10^{-1}$	6226.11
9	111202	65049	176251	-1.3650$\times10^{-10}$	5.7758$\times10^{-1}$	37571.28
10	301739	176506	478245	-4.9404$\times10^{-11}$	1.5392	271669.93
11	492276	287963	780239	-2.9730$\times10^{-11}$	2.4653	709903.47
12	682813	399420	1082233	-2.1035$\times10^{-11}$	3.3559	1340413.61

– **Theorem on the Records of** $\mathrm{VMax}(n)$**.**

Théorème 2.5. *The record values of the function $n \mapsto \mathrm{VMax}(n)$ occur precisely at indices n such that $n+1 = m+d$, where $\frac{m}{d}$ is an upper approximation of $X = \frac{\ln 2}{\ln 3 - \ln 2}$ obtained through the Stern–Brocot tree.*

Proof. The proof may be regarded as a direct adaptation of the argument used in the case $b > 0$, replacing lower approximations of X by their upper counterparts, which are more relevant in this context. The structural argument relies on the same growth mechanisms of $\mathrm{VMax}(n)$, and a parallel reasoning could be followed.
However, the focus here is on a distinct and essential phenomenon: the appearance, in numerical experiments, of reducible fractions equal to upper approximations of X. While the Stern–Brocot tree generates only irreducible approximants, these multiple fractions naturally arise through repetition of patterns and must be understood, justified, and then dismissed as redundant within the structural analysis of cycles.
The remainder of the proof aims to explain this phenomenon based on the transition patterns observed in the structure of the list $\mathrm{Ceil}(n)$, and to rigorously justify why only the irreducible upper approximants of X should be retained.
In the proof of the classical Syracuse conjecture (for $b = 1$), it was shown that the transition list $\mathrm{Ceil}(n)$ contains recurrent motifs—identical transition blocks repeated several times. These motifs appear for lengths N_1 corresponding to (lower) approximations of X.
For example, for $N_1 = 1054$, the list $\mathrm{Ceil}(N_1)$ contains a structuring motif that repeats for multiples $N_0 = kN_1$ with $1 < k \leq p$. Indeed, when the associated fraction remains constant, we have:

$$X - \frac{m_0}{d_0} = X - \frac{m_1}{d_1},$$

and denoting by R_k the associated remainders, we deduce:

$$\frac{\mathrm{VMax}(N_0 - 1)}{\mathrm{VMax}(N_1 - 1)} = \frac{R_0}{R_1}.$$

Using the formula obtained in Section 4.5, we find:

$$R_0 = \frac{1 - (F_1)^k}{1 - F_1} \cdot R_1,$$

where F_1 is a factor related to the geometric growth of the sequence.
Since $F_1 > 1$ (as we are considering an upper approximation), it follows that

$$(F_1)^k > F_1 \quad \Rightarrow \quad 1 - (F_1)^k < 1 - F_1 \quad \Rightarrow \quad \frac{1 - (F_1)^k}{1 - F_1} > 1,$$

and therefore:

$$\mathrm{VMax}(N_0 - 1) > \mathrm{VMax}(N_1 - 1).$$

Thus, every multiple of an upper approximation yields a larger value of $\mathrm{VMax}(n)$ and should theoretically appear in the list of records.

However, in practice, some corresponding rows may be missing from the numerical results due to rounding errors or limitations in arithmetic precision. Increasing the computational precision allows these cases to reappear.

Nonetheless, such cases may be ignored for two reasons:

* First, the values $\mathrm{VMax}(kN_1 - 1)$ and $\mathrm{VMax}(N_1 - 1)$ are very close, offering no significant gain in the analysis.
* Second, if a cycle of length N_1 exists, then its multiples kN_1 do not correspond to minimal cycles. The goal is to identify the fundamental structures.

This establishes that the indices n corresponding to records of $\mathrm{VMax}(n)$ coincide exactly with the irreducible upper approximations of X, thereby revealing the rational structure underlying the extremal behavior of the sequence. $\square$

- The constant c_α becomes c'_α in the case $b < 0$, since the maximal values $\mathrm{VMax}(n)$ are scaled by the factor $|b|$.
For $b = 1$, based on previous results, it can be shown that for $\alpha > 22$ (sufficient provided that the Central Limit Theorem is applied), a constant $c_\alpha = \lceil 945\alpha \rceil$ suffices to bound the size of the random lists required to guarantee the existence of at least $p + 1$ solutions.
In the general case $b < 0$, the growth of $\mathrm{VMax}(n)$ necessitates the introduction of an adjusted index $\alpha' = \alpha + \log_2(|b|)$, since multiplication by $|b|$ translates, in base-2 logarithmic terms, into an additive shift.
From this, we deduce:
$$c'_\alpha = 945\alpha > 945(\alpha' - \log_2 |b|).$$
Given the assumption that $|b| < 2^{10}$, we have $\log_2 |b| < 10$, so
$$c'_\alpha > 945(\alpha' - 10).$$
Moreover, since $\alpha' > 22$, we obtain:
$$c'_\alpha > 945 \left(\alpha' - \frac{10\alpha'}{22} \right) = 515\alpha'.$$
Likewise, given that $\alpha' > 48$, it follows that:
$$c'_\alpha > 1{,}382{,}371 \left(\alpha' - \frac{10\alpha'}{48} \right) > 1{,}094{,}377\, \alpha'.$$
This explicit bound allows all the asymptotic properties used in the proof to be preserved, particularly in the analysis of the parameter e in the Random List Theorem.

- **Testing the Conjecture up to $V_0 \le 2^\alpha |b|$**
For each value $b < 0$, we test the conjecture for all initial values $v_0 < 2^\alpha |b|$, with $\alpha = 22$. In every case, we observe the emergence of a very simple cycle, referred to as a *very trivial cycle*, defined as follows:
  - For every $b < 0$, choosing $v_0 = -b$ (which is odd) immediately yields a cycle of length 1:
$$v_1 = \frac{3v_0 + b}{2} = \frac{-3b + b}{2} = -b = v_0.$$
  - We also identify two families of short recurrent cycles:
    * If $b = -3(2k + 1)$ for some $k > 0$, then $v_0 = \frac{-b}{3} = -(2k + 1)$ is odd and leads to $v_1 = 0$, hence $v_n = 0$ for all $n \ge 1$: this is an absorbing cycle.
    * If $b = -9(2k+1)$ for some $k > 0$, then $v_0 = \frac{-b}{9} = 2k+1$ leads to a 2-cycle that passes through b:
$$\begin{aligned} v_1 &= \frac{3v_0 + b}{2} = \frac{-b + b}{2} = 0, \\ v_2 &= \frac{3v_1 + b}{2} = \frac{b}{2}, \\ v_3 &= \frac{3v_2 + b}{2} = \frac{3b + b}{2} = 2b, \quad v_4 = \frac{2b}{2} = b = v_2. \end{aligned}$$
This results in a 2-cycle with minimum value b.

Systematic tests were carried out for all values $-1024 < b < 0$, with $v_0 < -2^{22}b$.
The numerical results are as follows:

- 256 values of b admit only the very trivial cycle.
- The longest detected cycle has length 426, occurring for $b = -563$ and $v_0 = -19$.
- A total of 3927 "trivial" cycles were identified.

Elimination of Specific Transition Lists.
As in the case $b > 0$, we must exclude transition lists that reach a cycle from an initial value $W_0 < v_0$, where v_0 is the minimum of the cycle. Such a list $\mathrm{L}(N, m, d)$ consists of an initial access phase followed by several (possibly fractional) repetitions of the transition pattern of the cycle.

For instance, the list consisting solely of "1" transitions (which corresponds to the very trivial cycle) must be excluded.

These tests were performed over the same range of values for b. For each $b < 0$, we verify whether a trivial cycle can be reached from a value strictly smaller than its minimum. This ensures that $v_n > v$ for all n, justifying the exclusion.

The numerical results are available online [2], and enumerate all such cases, which remain scarce for each value of b.

- **Recurrence Step.**

  We apply the Random List Theorem with $\alpha > 22$ and $p = 230$. Let $n = \alpha + p$, $N = \lceil 515\alpha \rceil$, and assume
  $$f(N) < 1.2\,p + 0.95\,N.$$
  then we compute:
  $$e = n - N + f(N),$$
  $$e < 2.2\,p - 24\,\alpha.$$
  For $\alpha \geq 22$, this yields $e < -22 \ll 6$. According to the theorem, using the TCL part, this means that no more than 96 solutions can be found, while at least 231 are expected. This contradiction confirms the recurrence property.

  Remarque 2.6. For $\alpha > 48$, let $p = 1{,}190{,}000$, $n = \alpha + p$ and $N = \lceil 1{,}094{,}377\alpha \rceil$.
  Then $e < 2.2\,p - 54{,}717\,\alpha < -8{,}416 \ll 20$.
  This leads to a contradiction under the Berry–Esseen inequality, as we have $1{,}190{,}000 \gg 1{,}052{,}383$.

□

3 The $5n + 1$ Extension

3.1 Definitions

3.1.1 Extended Reduced Syracuse Sequence: V

Let $V_0 > 0$ be an initial integer. The sequence $(V_n)_{n \geq 0}$ is defined recursively by the rule:

$$V_{n+1} = \begin{cases} V_n/2 & \text{if } V_n \text{ is even} \quad \text{(type 0 transition)}, \\ (5V_n + 1)/2 & \text{if } V_n \text{ is odd} \quad \text{(type 1 transition)}. \end{cases}$$

This sequence defines a natural extension of the classical Syracuse sequence. We may also write $V_n = T^{(n)}(V_0)$ to emphasize the iteration of the associated transformation T.

3.1.2 Approximated Sequence: V'

We define the sequence $(V'_n)_{n \geq 0}$ by:

$$V'_{n+1} = \begin{cases} V'_n/2 & \text{if } V_n \text{ is even}, \\ 5V'_n/2 & \text{if } V_n \text{ is odd}, \end{cases} \qquad \text{with } V'_0 = V_0.$$

This defines a simplified approximation of V_n, in which the constant term $+1$ in odd transitions is neglected.

We then define the residue term R_n by

$$V_n = V'_n + R_n,$$

which will be analyzed in greater detail in the following sections.

3.2 Theorem for the $5n + 1$ Extension

The sequence (V_n) defined by the $5n+1$ extension exhibits behavior fundamentally different from the classical Syracuse case. The following result summarizes its main properties:

Théorème 3.1. *There are only finitely many cycles for the sequence defined by the $5n + 1$ extension, all of which can be detected by testing initial values $V_0 \leq 2^{23} = 8{,}388{,}608$. Moreover, there exists at least one value V_0 for which the sequence diverges.*

Experimental Evidence. Numerical tests were performed for all initial values $V_0 \leq 2^{20}$. To accelerate computation, a stopping criterion was introduced: the sequence is considered divergent if, for some $n > 800$, one has

$$V_n > \left(\frac{5}{2}\right)^{60} V_0.$$

Under this assumption, the following results were obtained:

- **4,174,834 values** of $V_0 \leq 2^{23}$ do not satisfy the conjecture under this criterion. The first and last such values are:

  $$7,\ 9,\ 11,\ 21,\ \ldots,\ 2015433,\ 2015435,\ 2015437,\ 2015439,\ 2015441,\ 2015443.$$

  This number is close to $2^{23}/2 = 4{,}194{,}304$, suggesting that only about 19,500 initial integers below 2^{23} yield a convergent orbit.
- Three finite cycles were identified, with minimal values 1, 13, and 17, and respective lengths 5, 7, and 7:

  $$\begin{aligned} &\{1,\ 3,\ 8,\ 4,\ 2\} && \text{(transitions: 11000)},\\ &\{13,\ 33,\ 83,\ 208,\ 104,\ 52,\ 26\} && \text{(transitions: 1110000)},\\ &\{17,\ 43,\ 108,\ 54,\ 27,\ 68,\ 34\} && \text{(transitions: 1100100)}.\end{aligned}$$
- It is worth noting that $V_0 = 5$ leads to an orbit reaching the cycle with minimum value 13. This trajectory will be excluded from the count when applying the theorem, although its impact is negligible in the asymptotic estimate.
- Similar observations hold for extensions of the form $5n + b$ with odd b: a finite number of cycles, all detectable under an explicit bound depending on b, and at least one initial value V_0 leading to a divergent orbit.

These results can be reproduced using the test files available online [2], associated with the $5n + b$ extensions.

Proof. The proof relies on an adaptation of the reasoning used in the classical $3n + 1$ case. However, two major structural differences must be taken into account:

1. The fundamental constant X that governs the growth estimates of trajectories is here replaced by

   $$X = \frac{\ln 2}{\ln 5 - \ln 2},$$

   due to the modified form of odd transitions.
2. The *Random List Theorem* is invoked in two distinct contexts:
   - to show that no cycles exist beyond those detected by testing all initial values $V_0 \leq 2^{23}$;
   - to prove the existence of at least one initial value V_0 that leads to a divergent orbit.

Since the general argument closely mirrors that of the $3n + 1$ case, we focus here only on the element that is truly specific to this extension: the lower bound on the number of candidate transition lists in the divergent case.

3.2.1 Steps Analogous to the Standard Proof

We follow, with appropriate adaptations, the main preliminary steps of the proof in the standard $3n + 1$ case [1]. In particular:

- The *Random List Theorem* generalizes without difficulty to extensions of the form $5n+b$ with b odd, and thus applies in particular to the $5n + 1$ case considered here. All underlying probabilistic assumptions remain valid.
- The analytical properties concerning the approximated sequence V' and the residue term R_n remain applicable, with the modification that the constant 3 in the odd transitions is replaced by 5. In particular, the bounds on R_n and its asymptotic behavior retain the same qualitative structure.
- The transition list $\mathrm{Ceil}(N)$, defined by $m_n = \left\lceil \frac{\ln 2}{\ln 5} \cdot n \right\rceil$, plays the same fundamental role. It exhibits characteristic patterns related to the upper approximations of the constant

  $$X = \frac{\ln 2}{\ln 5 - \ln 2}.$$

  These patterns are visible in the associated numerical simulations. For example, for $(N, m, d) = (339, 146, 193)$, one recovers the usual structural inequalities:

  $$\frac{m_n}{10} < R_n < m_n,$$

  which suffice to ensure the validity of the combinatorial arguments used in the remainder of the proof.

- The construction of the boundary $\mathrm{JGL}(N, v_0)$ follows exactly the same procedure as in the $3n+1$ case. The characterization of record values once again relies on the lower approximations of the constant X. The minimal lengths c_α of cycles for $v_0 = 2^\alpha$ thus satisfy the inequality:

$$c_\alpha > \lceil 800\,\alpha \rceil \quad \text{for all } \alpha \geq 23.$$

These results are supported by the following numerical data, extracted from simulations related to the $5n+1$ extension. They reveal a rapid growth in the maximum orbit size compatible with the transition constraints. Let $c = n \cdot \ln 2 / \ln(\lfloor \mathrm{VMax}(n) \rfloor)$:

k	m	d	n	$\mathrm{VMax}(n)$	c
1	1	1	2	$3.33 \approx 2^{-1.585}$	0
2	2	2	4	$1.29 \approx 2^{0.3626}$	∞
3	3	3	6	$20.33 \approx 2^{4.346}$	1.388
4	31	40	71	$322.49 \approx 2^{8.333}$	8.522
5	59	77	136	$1978.73 \approx 2^{10.950}$	12.420
6	205	270	475	$9015.84 \approx 2^{13.138}$	36.154
7	351	463	814	$22579.49 \approx 2^{14.463}$	56.283
8	497	656	1153	$59600.21 \approx 2^{15.863}$	72.685
9	643	849	1492	$569835.42 \approx 2^{19.120}$	78.033
10	4647	6142	10789	$6.76 \times 10^6 \approx 2^{22.689}$	475.519
11	8651	11435	20086	$3.52 \times 10^7 \approx 2^{25.070}$	801.203
12	21306	28164	49470	$4.29 \times 10^8 \approx 2^{28.677}$	1725.051
13	97879	129388	227267	$3.92 \times 10^{10} \approx 2^{35.190}$	6458.340
14	1936274	2559614	4495888	$5.98 \times 10^{12} \approx 2^{42.444}$	105925.051

These data provide empirical confirmation that the JGL lists for the $5n+1$ extension behave analogously to those in the $3n+1$ case, with appropriately adjusted constants.

3.2.2 Nonexistence of Additional Cycles

Proof. We have already established the following facts:

- If $V_0 = 2^\alpha$ is the minimal value of a hypothetical cycle, and lies within the interval

$$]\mathrm{VMax}(\mathrm{NX}_{k-1} - 1),\, \mathrm{VMax}(\mathrm{NX}_k - 1)]\,,$$

then the minimal length N of the cycle satisfies:

$$N = \mathrm{NX}_k > 800\,\alpha, \quad \text{for } \alpha \geq 23.$$

- The associated transition list, denoted $\mathrm{L}(N, m, d)$, satisfies:

$$m = \left\lfloor \frac{\ln 2}{\ln 5} \cdot N \right\rfloor,$$

and all elements of the cycle correspond to circular permutations sharing the same global characteristics (N, m, d).

We now study the set $\mathrm{Cy}(N, V_0)$ of transition lists $\mathrm{L}(N, m, d)$ such that V_0 is the minimum of the orbit among the first N terms (i.e., $V_n \geq V_0$ for all $0 \leq n \leq N$), with m defined as above.

Using the inequality $V_{n+1} \leq (5V_n+1)/2 \leq 2^2 V_n$, an inductive argument analogous to that of the standard case yields two key properties for the first $p+1$ terms $V_0, V_1, \ldots, V_p$:

- Each V_i is a solution of a transition list in $\mathrm{Cy}_{2p}(N, V_0)$, which means a list that is greater than $\mathrm{JGL}_{2p}(N, V_0)$ in the partial order defined on transition lists, with

$$m = \left\lfloor \frac{\ln 2}{\ln 5}(N - 2p) \right\rfloor.$$

- Each V_i belongs to the interval:

$$V_i \in \left[V_0,\, 2^{2p} V_0\right[.$$

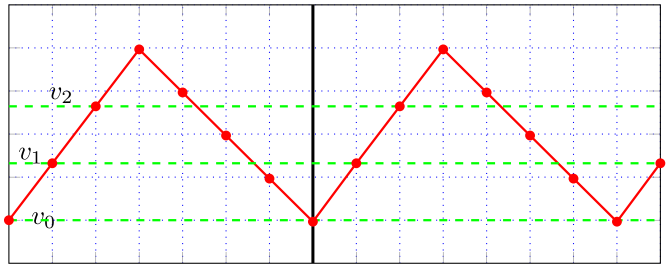

Figure 2: Cyclic visualization of the orbit with reference to the minimal value V_0

Remarks.

- By considering the final terms of the cycle, it is easy to show that *at least* $3p+1$ values of the orbit lie within the interval $\left[V_0,\, 2^{2p}V_0\right]$ (for example, $V_{N-1} \le 2V_0$, $V_{N-2} \le 2^2V_0$, etc.). This provides additional margin in the application of the Random List Theorem.
- A similar strategy could be used in the standard $3n+1$ case, likely with $4p+1$ terms within the same interval.

This diagram illustrates the structure of a cycle seen as a circular permutation from each V_n, and shows that the first $p+1$ terms remain within a controlled zone.

Application of the Theorem. To conclude, we apply the Random List Theorem with:

$$n = \alpha + 2p, \quad N = c_\alpha = \lceil 800\alpha \rceil, \quad \text{for } \alpha > 23,\ p = 100.$$

We must bound the number of admissible lists $\mathrm{L}(N, m, d)$, where m lies in the interval:

$$\left\lfloor \frac{\ln 2}{\ln 5}(N-2p) \right\rfloor \le m \le \left\lfloor \frac{\ln 2}{\ln 5} N \right\rfloor.$$

A rough upper bound is obtained by:

$$\text{nb} < \frac{N}{2} \cdot \binom{N}{m}, \quad \text{with } m = \left\lfloor \frac{\ln 2}{\ln 5} N \right\rfloor, \text{ since } \frac{\ln 2}{\ln 5} < \frac{1}{2}.$$

Using Stirling's formula (as in the standard case), we get:

$$\binom{N}{kN} \approx \frac{1}{\sqrt{2\pi k(1-k)N}} \cdot \left(\frac{1}{k^k (1-k)^{1-k}} \right)^N, \quad \text{with } k = \frac{\ln 2}{\ln 5}.$$

Taking base-2 logarithms:

$$\begin{aligned} f(N) <\ & \log_2 N - 1 - \frac{1}{2}\log_2(2\pi k(1-k)) \\ & - \frac{1}{2}\log_2 N - N \cdot \left(k \log_2 k + (1-k)\log_2(1-k)\right). \end{aligned}$$

This yields the estimate:

$$f(N) < 0.98609\, N + \tfrac{1}{2}\log_2 N - 1.216.$$

Finally, we evaluate $e = n - N + f(N)$ with $p = 110$:

$$e < -10.128\,\alpha + \tfrac{1}{2}\log_2 \alpha + 219.654,$$

which is strictly decreasing for $\alpha > 23$.

$$e < -7 < 6 \quad \text{for } \alpha > 23.$$

Conclusion. We should have at least $p+1 = 111$ minimal solutions, yet we have shown that the total number is less than 96 with the Central Limit Theorem version of the *Random List Theorem*. This contradiction rules out the existence of a cycle with $V_0 = 2^\alpha$ as its minimal value. The recurrence hypothesis is therefore verified, completing the proof.

Remarque 3.2. If needed, one can easily take $\alpha_0 > 23$—for example, $\alpha_0 = 48$—which is large enough to enable the use of the Berry–Esseen inequality.

□

3.2.3 Existence of a Value for Which the Sequence Diverges

Proof. As in the previous cases, our goal is to apply the *Random List Theorem* by counting the elements of the set $\mathrm{Up}(N, V_0)$, defined as the set of transition lists of length N such that V_0 is the minimum value over the orbit restricted to the first N terms.

The illustration below visualizes the different types of paths:

- The blue curve represents the boundary $\mathrm{JGL}(N, V_0)$.
- The green line corresponds to a path (transition list) that stays entirely above the JGL boundary: it is therefore admissible.
- The black line represents the Catalan triangle boundary.
- Green points indicate the endpoints of valid paths (those remaining above the JGL boundary).
- Black points indicate invalid paths that cross below the boundary.

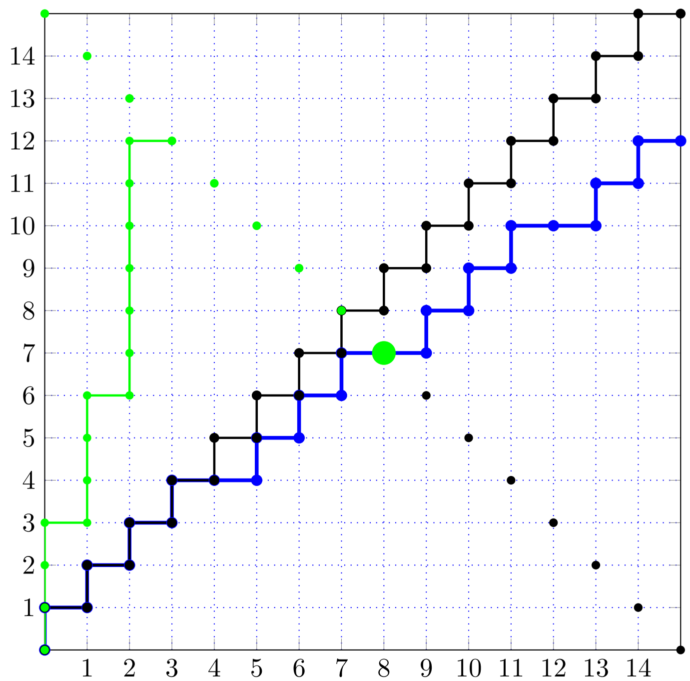

Figure 3: Valid and invalid paths in the transition triangle.

We show that:

- If the transition t_N in the list $\mathrm{JGL}(N, V_0)$ is of type 0, then
$$\mathrm{Up}(N+1, V_0) = 2 \cdot \mathrm{Up}(N, V_0).$$
- If t_N is of type 1, then:
$$\mathrm{Up}(N+1, V_0) = 2 \cdot \mathrm{Up}(N, V_0) - \mathrm{Bo}(N),$$
where $\mathrm{Bo}(N)$ denotes the number of paths of length N that reach the boundary JGL exactly at step N.

As long as the boundary remains vertical ($t_N = 1$), the quantity $\mathrm{Up}(N, V_0)$ is increasing: we have $\mathrm{Up}(N, V_0) = 1$, then $\mathrm{Up}(N+1, V_0) > \mathrm{Up}(N, V_0)$. Thus, the function is monotonic.

The value $\mathrm{Up}(N, V_0)$ can be approximated by 2^{N-p}, where p depends on the parameter a of the Syracuse extension (i.e., $(an+1)/2$).

Numerical Example for $a = 5$: This corresponds to an effective slope of $\lambda = \frac{\ln 2}{\ln 5} \approx 0.43$, measuring the proportion of type 1 transitions.

N	p	$\mathrm{Up}(N) \approx 2^{N-p}$
100	2.4691	$2^{97.5309}$
200	2.4994	$2^{197.5006}$
500	2.5060	$2^{497.4940}$
1000	2.5061	$2^{997.4939}$

We observe that p converges to a constant as N increases, confirming that $\mathrm{Up}(N, V_0)$ grows exponentially and that the density of valid paths remains significant.

Heuristic Observation: The asymptotic behavior appears to vary depending on the slope of the boundary:

- If $\lambda < 0.48$, the growth of $\mathrm{Up}(N)$ is exponential (and p stabilizes);
- If $\lambda > 0.50$, p appears to diverge rapidly, significantly reducing the number of valid paths.

Constructive Lower Bound Method: We now aim to provide an explicit lower bound for $\mathrm{Up}(N)$. We approximate the real boundary (with irrational slope λ) by a periodic boundary associated with a rational approximation of $\lambda = \frac{\ln 2}{\ln 5}$.

We use the upper rational approximation $\lambda \lesssim \frac{4}{9} \approx 0.444$, corresponding to a transition pattern of length 9 with 4 type 1 transitions.

This pattern, although simpler than the actual boundary, always lies above it, so any admissible list under the approximate boundary is also admissible for the true boundary.

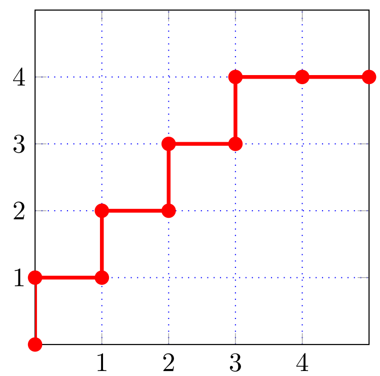

Figure 4: Elementary 5×4 pattern used to approximate the boundary.

This elementary motif allows approximation of the true JGL boundary using a simple periodic form, facilitating exact path count estimation.

The following figure illustrates the juxtaposition of the red motif (periodic approximation) with the actual boundary (in blue). The black diagonal is used to frame the regions resembling a "Catalan trapezoid".

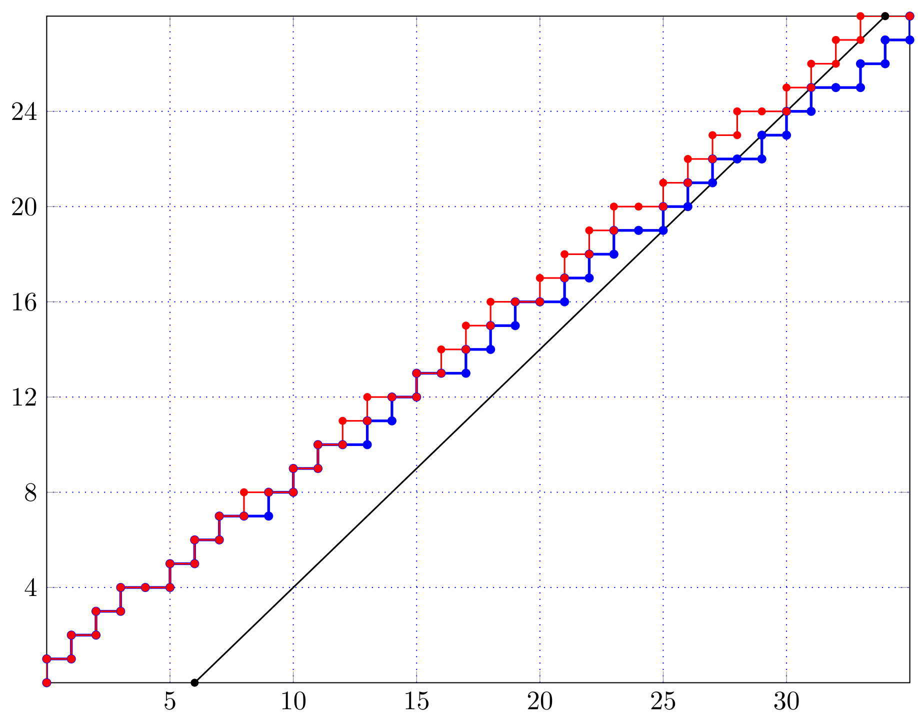

Figure 5: True boundary (blue), 5×4 motif approximation (red), and reference diagonal (black).

Estimating the Number of Admissible Paths via a Periodic Motif. Let U_N denote the number of transition lists of length N that stay above the boundary approximated by a 4×5 rectangular motif, corresponding to a slope of $\frac{4}{9}$.

We focus on the subsequence U_{9k}, corresponding to lengths that are multiples of 9, i.e., to the concatenation of k motifs.

Assume that from some index k_0 onward, we have:

$$\mathrm{U}_{9(k+1)} > 2^{9-e_k} \cdot \mathrm{U}_{9k} \quad \text{and} \quad \mathrm{U}_{9k_0} = 2^{9k_0 - l_0}.$$

Then for all $t > k_0$:

$$\mathrm{U}_{9t} > \prod_{k=k_0}^{t-1} 2^{9-e_k} \cdot 2^{9k_0 - l_0} = 2^{9t - g(t)} \quad \text{where} \quad g(t) := l_0 + \sum_{k=k_0}^{t-1} e_k.$$

If $g(t)$ admits a finite limit l as $t \to \infty$, then we obtain the asymptotic lower bound:

$$\mathrm{U}_{9k} > 2^{9k-l}, \quad \text{and hence} \quad \mathrm{Up}(9k, V_0) > 2^{9k-l}.$$

Analysis of the Recurrence for U_{9k}. Let $\mathrm{A}_{k,n}$ denote the number of paths reaching the point with coordinates $(5k+n, 4k+n)$ on the boundary. Then we have:

$$\begin{aligned}
\mathrm{U}_{9k+1} &= 2\,\mathrm{U}_{9k} - \mathrm{A}_{k,0}, \\
\mathrm{U}_{9k+2} &= 2\,\mathrm{U}_{9k+1} = 2^2\,\mathrm{U}_{9k} - 2\,\mathrm{A}_{k,0}, \\
\mathrm{U}_{9k+3} &= 2\,\mathrm{U}_{9k+2} - \mathrm{A}_{k,1} = 2^3\,\mathrm{U}_{9k} - 4\,\mathrm{A}_{k,0} - \mathrm{A}_{k,1}, \\
\mathrm{U}_{9k+4} &= 2\,\mathrm{U}_{9k+3} = 2^4\,\mathrm{U}_{9k} - 8\,\mathrm{A}_{k,0} - 2\,\mathrm{A}_{k,1}, \\
\mathrm{U}_{9k+5} &= 2\,\mathrm{U}_{9k+4} - \mathrm{A}_{k,2} = 2^5\,\mathrm{U}_{9k} - 16\,\mathrm{A}_{k,0} - 4\,\mathrm{A}_{k,1} - \mathrm{A}_{k,2}, \\
\mathrm{U}_{9k+6} &= 2\,\mathrm{U}_{9k+5} = 2^6\,\mathrm{U}_{9k} - 32\,\mathrm{A}_{k,0} - 8\,\mathrm{A}_{k,1} - 2\,\mathrm{A}_{k,2}, \\
\mathrm{U}_{9k+7} &= 2\,\mathrm{U}_{9k+6} - \mathrm{A}_{k,3} = 2^7\,\mathrm{U}_{9k} - 64\,\mathrm{A}_{k,0} - 16\,\mathrm{A}_{k,1} - 4\,\mathrm{A}_{k,2} - \mathrm{A}_{k,3}, \\
\mathrm{U}_{9k+8} &= 2\,\mathrm{U}_{9k+7} = 2^8\,\mathrm{U}_{9k} - 128\,\mathrm{A}_{k,0} - 32\,\mathrm{A}_{k,1} - 8\,\mathrm{A}_{k,2} - 2\,\mathrm{A}_{k,3}, \\
\mathrm{U}_{9(k+1)} = \mathrm{U}_{9k+9} &= 2\,\mathrm{U}_{9k+8} = 2^9\,\mathrm{U}_{9k} - 256\,\mathrm{A}_{k,0} - 64\,\mathrm{A}_{k,1} - 16\,\mathrm{A}_{k,2} - 4\,\mathrm{A}_{k,3}.
\end{aligned}$$

Since $\mathrm{A}_{k,0} \le \mathrm{A}_{k,1} \le \mathrm{A}_{k,2} \le \mathrm{A}_{k,3}$, we obtain the inequality:

$$\mathrm{U}_{9(k+1)} \ge 2^9\,\mathrm{U}_{9k} - 340\,\mathrm{A}_{k,3}.$$

Expressing the quantity e_k. Let us define:

$$2^9 \cdot \mathrm{U}_{9k} - 340 \cdot \mathrm{A}_{k,3} = 2^{9-\mathrm{e}_k} \cdot \mathrm{U}_{9k}.$$

This is equivalent to:

$$2^9 \cdot \mathrm{U}_{9k} \left(1 - \frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}}\right) = 2^9 \cdot \mathrm{U}_{9k} \cdot 2^{-\mathrm{e}_k},$$

and therefore:

$$1 - \frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}} = 2^{-\mathrm{e}_k}.$$

Taking natural logarithms gives:

$$\ln\left(1 - \frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}}\right) = -\mathrm{e}_k \cdot \ln 2.$$

This expression is well-defined as long as the ratio $\frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}} < 1$, which we will justify later.

We now use the classical approximation:

$$\ln(1+u) \sim u \quad \text{as } u \to 0,$$

which yields:

$$\mathrm{e}_k \approx \frac{1}{\ln 2} \cdot \frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}} \approx \frac{0.958 \cdot \mathrm{A}_{k,3}}{\mathrm{U}_{9k}}.$$

In particular, we obtain the useful upper bound:

$$\mathrm{e}_k < \frac{\mathrm{A}_{k,3}}{\mathrm{U}_{9k}}.$$

Upper Bound on $\mathrm{A}_{k,3}$. We use an inequality derived from the Catalan trapezoid [4]:

$$\mathrm{A}_{k,3} < \binom{9k+6}{5k+3} - \binom{9k+6}{4k+2}.$$

Neglecting the second term, we get a crude upper bound:

$$\mathrm{A}_{k,3} < \binom{9k+6}{5k+3}.$$

Using the approximation $\ln(n!) \approx n \ln n - n$, we derive:

$$\ln \mathrm{A}_{k,3} < (9k+6)\ln(9) - (5k+3)\ln(5) - (4k+3)\ln(4) \approx 6.183k + 4.2,$$

so that:

$$\mathrm{A}_{k,3} < 2^{8.921k+6.06}.$$

Lower Bound on U_{9k}. Since the boundary with slope $\frac{4}{9}$ lies below the diagonal (slope $\frac{1}{2}$), we have:

$$\mathrm{U}_{9k} > \binom{9k}{4.5k}.$$

Using Stirling's formula, this gives:

$$\mathrm{U}_{9k} > \frac{\sqrt{2}}{\sqrt{9\pi k}} \cdot 2^{9k - \frac{\ln k}{2\ln 2}}.$$

Conclusion on e_k and A Posteriori Justification. We can now return to e_k and justify the previous approximation $\ln(1+u) \sim u$ by showing that:

$$\frac{340 \cdot \mathrm{A}_{k,3}}{2^9 \cdot \mathrm{U}_{9k}} < 2^{8.921k+6.06-\left(9k - \frac{\ln k}{2\ln 2}\right)} = 2^{-0.079k + \frac{\ln k}{2\ln 2} + 6.06},$$

which tends to 0 as $k \to \infty$. Thus, the use of the approximation is justified.

We then obtain:

$$\mathrm{e}_k < 2^{-0.079k + \frac{\ln k}{2\ln 2} + 6.06},$$

which for $k \geq 1000$ yields:

$$\mathrm{e}_k < 2^{-0.079k},$$

i.e., a geometric sequence with ratio strictly less than 1.

Consequence: the series $\sum \mathrm{e}_k$ converges, and the function $g(9k) = l_0 + \sum_{j=k_0}^{k-1} \mathrm{e}_j$ has a finite limit. Choosing $k_0 = 1000$, we obtain:

$$\mathrm{U}_{9k} > 2^{9k - l_0} \quad \text{with } l_0 = 2.69258, \qquad \sum_{k \geq 1000} \mathrm{e}_k < 10^{-9}.$$

Therefore:

$$\mathrm{Up}(9k, V_0) > 2^{9k-2.7}.$$

By using explicit recurrence relations between U_{9k+i} and U_{9k} for $0 < i < 9$, we extend the bound to all N:

$$\boxed{\mathrm{Up}(N, V_0) > 2^{N-2.7} \quad \text{for all } N.}$$

Summary. For every $\mathrm{N} \geq 23$, let $\mathrm{E_N}$ be the set of initial values $V_0 < 2^{23} \leq 2^{\mathrm{N}}$ such that V_0 is the minimum value along some transition list of length N, meaning $V_n \geq V_0$ for all $1 \leq n \leq \mathrm{N}$. This condition ensures that V_0 is the true minimum over its restricted orbit (we consider $b = 1$, so $V_0 > 0$).

We apply the *Random List Theorem* with $n = 23$ and $f(\mathrm{N}) = \mathrm{N} - 3$, yielding:

$$e = n - \mathrm{N} + f(\mathrm{N}) > 23 - \mathrm{N} + \mathrm{N} - 3 = 20.$$

Therefore, by applying the Berry–Esseen inequality, we deduce that there are more than $520{,}481 > 2^{18}$ solutions, and thus:

$$2^{18} < \#\mathrm{E_N} < 2^{\mathrm{N}}.$$

By construction, we have the inclusion $\mathrm{E_N} \subset \mathrm{E_P}$ for all $\mathrm{N} > \mathrm{P}$, since the orbit condition becomes more restrictive as the length increases.

Fix $\mathrm{N}_1 = 23$. We now show by contradiction that there exists at least one $V_0 \in \mathrm{E}_{\mathrm{N}_1}$ such that $V_0 \in \mathrm{E_N}$ for infinitely many values of N.

Suppose the contrary: for every $e \in \mathrm{E}_{\mathrm{N}_1}$, only finitely many integers N satisfy $e \in \mathrm{E}_\mathrm{N}$. Let

$$M_e = \max\{\mathrm{N} \mid e \in \mathrm{E}_\mathrm{N}\}, \quad M_{\mathrm{E}_1} = \max_{e \in \mathrm{E}_{\mathrm{N}_1}} M_e.$$

Now take $\mathrm{N} = M_{\mathrm{E}_1} + 1$. By hypothesis, no element of $\mathrm{E}_{\mathrm{N}_1}$ can belong to E_N, so:

$$\mathrm{E}_\mathrm{N} \subset \mathrm{E}_{\mathrm{N}_1} \quad \text{and} \quad \mathrm{E}_\mathrm{N} = \emptyset,$$

which contradicts the fact that $\#\mathrm{E}_\mathrm{N} > 2^{18}$. Therefore, the assumption is false, and there exists at least one $V_0 \in \mathrm{E}_{\mathrm{N}_1}$ that belongs to infinitely many sets E_N.

Such a value V_0 cannot yield a nontrivial cycle. Indeed, if the orbit of V_0 formed a cycle from some index n_0, then $V_{n_0} = V_{n_1}$ for some $n_1 < n_0$, and the minimality of V_0 would force this equality to occur for a fixed value of N, contradicting its membership in infinitely many E_N.

(We disregard here the rare cases where a trivial cycle is reached from an initial value smaller than the cycle's minimum; such cases do not arise for $b = 1$.)

We now show, again by contradiction, that the sequence V_n diverges from this value V_0. Suppose it were bounded, i.e., there exists an upper bound M such that $V_n < M$ for all n. Since $V_0 \in \mathrm{E}_\mathrm{N}$ for infinitely many N and $V_n \geq V_0$, the values V_n would all be distinct (otherwise a cycle would occur). This would imply infinitely many distinct values in the finite interval $[V_0, M[$, which is impossible.

Conclusion. There exists at least one initial value V_0 such that the sequence V_n (and hence U_n) diverges. For instance, $V_0 = 7$ is a plausible candidate.

- **For $a = 5$ and $b = 1$, there exists at least one value V_0 for which the sequence V_n diverges.**

The same reasoning can be reproduced for any other odd integer b, with only trivial cycles requiring adjustment.

□

Conclusion

The results presented in this work confirm that the dynamical behavior of the $5n + 1$ extension differs fundamentally from that of the classical $3n + 1$ case, while still exhibiting sufficient structural regularity to allow a rigorous analysis using the Random List Theorem.

Two major points have been established:

- The number of nontrivial cycles is finite. All such cycles have been detected experimentally up to $V_0 \leq 2^{23}$, and their nonexistence beyond this range has been rigorously proven using a combinatorial method based on the enumeration of minimal transition lists.
- There exists at least one initial value V_0 for which the sequence diverges. This divergence is justified by the controlled growth in the number of valid paths above the JGL boundary, which itself is upper-bounded by a periodic motif whose behavior can be estimated analytically.

The methodological framework employed here is fully generalizable to other extensions of the form $an+b$, with appropriate adjustments to the fundamental constants and boundary structures. These results provide a solid foundation for the exploration of entire families of Syracuse-like extensions, paving the way toward a broader understanding of nonlinear discrete dynamics of Collatz type—such as those discussed, for instance, in response to arXiv:2107.11160v4 [3].

□

References

[1] J. Ballasi. *A Combinatorial Proof of the Syracuse Conjecture Using Transition Lists*, Zenodo, 2025.
`https://doi.org/10.5281/zenodo.15653126`

[2] J. Ballasi, *Interactive interface for testing properties of the Syracuse sequence*, 2025. (supplementary unpublished work)
`https://www.bajaxe.com/tests/index.html`

[3] J. Ballasi, *Answers to questions about the Syracuse sequence*, Zenodo, 2025. (supplementary unpublished work).
`https://doi.org/10.5281/zenodo.15653288`

[4] *Catalan's Triangle*. Wikipedia.
`https://en.wikipedia.org/wiki/Catalan%27s_triangle`