Extended Syracuse sequences (3*n*+1, 3*n*+*b*, 5*n*+1,...) : Proofs of the different conjectures

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Abstract

First, we will list a set of definitions, a practical summary to facilitate the reading of the document.

We will state the different theorems that we will prove in this document.

We will first study the extensions of the Syracuse sequence defined as follows : if u_n is odd then $u_{n+1} = 3u_n + b_1$ (with b_1 odd, $b_1 = 1$ for the standard case)
If, at first, we focus on the classical Syracuse conjecture, we can ignore the texts written in green which globally correspond to the extension $b_1 \neq 1$.

We will retrieve Shalom Eliahou's result (the link between the minimum length of a non-trivial cycle and the maximum value for which the conjecture has been verified) in another way, using a JGL boundary transition list, which will serve us for the following reasoning.
The section V, the most delicate and the longest, which mathematically proves this result observed by calculations, can be ignored at a first reading.

Thanks to a new Theorem, we will be able to prove that the elements of a non-trivial cycle are necessarily in an interval that we will specify.

We can then easily conclude that there are no non-trivial cycles, the lower bound of this interval being greater than the maximum possible value for having a cycle.
A similar reasoning will allow us to conclude that there is no divergence towards infinity.
The theoretical results of the method are consistent with the tests and also with the Eric Roosendaal's list of records found up to that date.

To verify the robustness of the method, we will study the case $u_{n+1} = 5u_n + 1$ and prove that there is at least one value for which the sequence diverges.

The proofs, which can be improved, only use fairly simple mathematical reasoning and can be understood in the smallest details by many interested people, with everyone certainly able to understand the main points.

The purpose of the document will be achieved, but, to go further, I have stated my own conjecture !

Note concerning the English part of the document :

I'm French-speaking and the English version of the document was obtained in "my" very approximate English or using the offline "Google Translate" application.
Knowing that online translation is certainly much better and given the very rapid evolution of the quality of translations in general, it is certainly better to launch the online translation of the French part of the document.

Note regarding PDF documents :

The HTML documents were written using the formal web editing tools I developed in the early 2000s (perhaps even before the projects that led to Mathjax).
The rendering has not been updated for over 15 years and browser characteristics have changed a bit in the meantime, therefore there are a few inaccuracies in the placement of the indices (which could be corrected)
Converting from HTML to PDF format to obtain a static document further degrades the quality (missing fraction bars) and increases the file size.
The proofs are quite short, about 70 pages. The results at the end of the file represent the largest part of the file, about 500 pages.

English | Français | Font-size : 1 | OK

Switch to the proof for the standard sequence

Choose the value of b_1 : [ ] OK (odd integer)

Switch to proof of the minimum length of a cycle

I. Definitions
   a. The Syracuse sequence (extended) : u
   b. The reduced (extended) Syracuse sequence : v
   c. Trajectory of length $N = m + d$ or transition list $L(N, m, d)$ for v:
   d. The approximated reduced Syracuse sequence : v'
   e. Notations
      - $X = \frac{\mathrm{Ln}2}{\mathrm{Ln}3 - \mathrm{Ln}2}$
      - $\mathrm{appX}(k) = \frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ the k^th defect approximation of X
      - Inequality (In-1)
      - $\mathrm{VMax}(n) = \frac{2^n r_n}{2^n - 3^{m_n}}$
      - $C_N^{m_N} = \binom{N}{m_N}$, old notation for the number of permutations
      - $B_1 = \lceil \frac{\mathrm{Ln}(|b_1|)}{\mathrm{Ln}2} \rceil$
      - $N_0 = 17$
      - $maxSyr\_b1 = 2^{B_1 + N_0}$

II. Theorems that will be proven in the document
   1. Extensions of the type $3n + b_1$
      - Theorem $3n + b_1$: For each odd value of b_1 such that $|b_1| < 1024$, u does not diverge, and the only cycles are the trivial cycles obtained by testing the Syracuse sequence for all $u_0 \le 2^{N_0 + B_1}$ with $N_0 = 17$ and $B_1 = \lceil \frac{\mathrm{Ln}(|b_1|)}{\mathrm{Ln}2} \rceil$
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   1. Purpose
   2. Approximated reduced Syracuse sequence : v'
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   4. Condition for the existence of a non-trivial cycle , for all $b_1 > 0$ and $v_0 > 0$
      - Necessary condition IV-4-1 : $\frac{r_N}{1 - \frac{3^m}{2^N}} \ge maxSyr\_b1$ to have a non-trivial cycle of length N, with v_0 the minimum value of the cycle

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I. **Definitions** :

We list a set of definitions, a summary or practical reminder to facilitate the reading of the document.

a. **The Syracuse sequence (extended) : *u***

We define $b_1 = 2b+1$, an odd integer because b can be negative.

$$\begin{cases} u_0 > 0 \\ u_{n+1} = \dfrac{u_n}{2} \text{ if } u_n \text{ is even (transition of type 0)} \\ u_{n+1} = 3u_n + b_1 \text{ if } u_n \text{ is odd (transition of type 1)} \end{cases}$$

b. **The reduced (extended) Syracuse sequence : *v***

As if u_n is odd, u_{n+1} is even by construction, it is interesting to make the following transition directly.
The reduced Syracuse sequence brings together this transition.

We define $b_1 = 2b+1$, an odd integer because b can be negative.

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \dfrac{3v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

c. **Trajectory of length $\mathrm{N} = m + d$ or transition list $\mathrm{L}(\mathrm{N}, m, d)$ for *v*:**

A trajectory of length N is a list of "type 0" or "type 1" transitions, thus a word composed of 0 and 1.

We will set d, the number "type 0" transitions and m the number "type 1" transitions and $\mathrm{N} = m + d$:

For example, with $b_1 = 1$:

With v, we get $v_0 = 7$ =="type 1"==> 11 =="type 1"==> 17 =="type 1"==> 26 =="type 0"==> 13, the trajectory is $\mathrm{L}(\mathrm{N}, m, d) =$ "1110" with $\mathrm{N} = 4$, $m = 3$ and $d = 1$

d. **The approximated reduced Syracuse sequence : v'**

Let's define the approximated sequence by replacing the term $3 v_n + b_1$ with $3 v_n$
It is immediate that the approximation makes sense for values of v_0 that are large enough and values of n that are low enough.

We ensure that the transitions of v' are identical to those of v

$$\begin{cases} v'_0 = v_0 > 0 \\ v'_{n+1} = \dfrac{v'_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v'_{n+1} = \dfrac{3 v'_n}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

We can notice that some elements of v' are not integers.

e. **Notations**

- The value $\boldsymbol{X} = \dfrac{\mathbf{Ln2}}{\mathbf{Ln3 - Ln2}}$ and therefore $\dfrac{X}{1+X} = \dfrac{\mathrm{Ln2}}{\mathrm{Ln3}}$, with $\mathrm{Ln}(n)$, the natural logarithm of n
- **appX(k) =** $\dfrac{\mathbf{mX}_k}{\mathbf{dX}_k}$ **the k**$^{\mathbf{th}}$ **defect approximation of X** (with $\dfrac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}} < 1$ and $\mathrm{NX}_k = \mathrm{mX}_k + \mathrm{dX}_k$)
- **Inequality (In-1)** : $\left(1 - \dfrac{3^{m_n}}{2^{n+1}}\right) v_0 \le \dfrac{r_n}{2}$
- **VMax(n) =** $\dfrac{2^n r_n}{2^n - 3^{m_n}}$, the maximum value of v_0 that satisfies this inequality (In-1)
- $\mathbf{C}^{m_{\mathrm{N}}}_{\mathbf{N}} = \dbinom{\mathbf{N}}{m_{\mathbf{N}}}$**, old notation for the number of permutations**
- $B_1 = \left\lceil \dfrac{\mathbf{Ln}(|b_1|)}{\mathbf{Ln2}} \right\rceil$
- $N_0 = 17$
- ***maxSyr_b1*** $= 2^{B_1 + N_0}$

II. **Theorems that will be proven in the document**

1. **Extensions of the type $3n + b_1$**
   We define the extended Syracuse sequence : u

   We define $b_1 = 2b + 1$, an odd integer because b can be negative.

$$\begin{cases} u_0 > 0 \\ u_{n+1} = \dfrac{u_n}{2} \text{ if } u_n \text{ is even (transition of type 0)} \\ u_{n+1} = 3 u_n + b_1 \text{ if } u_n \text{ is odd (transition of type 1)} \end{cases}$$

Remarks :

- Instead of extending the definition of u by taking $b_1 < 0$, we could have chosen to keep $b_1 > 0$ and extend the definition to $u_0 < 0$.
  Moreover, for $b_1 < 0$, if we define the sequence u' with $u'_0 = -u_0 < 0$ and $b_1' = -b_1$, then $u' = -u$ for $u'_0 < 0$ and $b_1 > 0$.
  Indeed, by immediate induction, this is true at rank 0 since $u'_0 = -u_0$ and if we assume that it is true at rank n, then we can easily obtain the property at the rank $n+1$:
  - If $u'_n = -u_n$ is even, then $u'_{n+1} = \frac{u'_n}{2} = \frac{-u_n}{2} = -u_{n+1}$
  - If $u'_n = -u_n$ is odd, then $u'_{n+1} = 3u'_n + b_1' = -3u_n - b_1 = -(3u_n + b_1) = -u_{n+1}$

  This equivalent way to see the extension, with $u_0 < 0$ instead of $b_1 < 0$, will be used in the proof as it simplifies the reasoning.
- In the document, to improve readability, we will not specify that u depends from b_1

We define the extended reduced Syracuse sequence : v
As if u_n is odd, u_{n+1} is even by construction, it is interesting to make the following transition directly.
The reduced Syracuse sequence brings together this transition.

We define $b_1 = 2b+1$, an odd integer because b can be negative.

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \frac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \frac{3v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

Remarks :
- In the document, to improve readability, we will not specify that v depends on b_1
- Equivalently, the extension could have been done with $v_0 < 0$ instead of $b_1 < 0$.
  This will be used in the proof as it simplifies the reasoning.
- We will systematically use the sequence v instead of u throughout the proof

Theorem $3n+b_1$: For each odd value of b_1 such that $|b_1| < 1024$, u does not diverge, and the only cycles are the trivial cycles obtained by testing the Syracuse sequence for all $u_0 \leq 2^{N_0+B_1}$ with $N_0 = 17$ and $B_1 = \lceil \frac{\text{Ln}(|b_1|)}{\text{Ln}2} \rceil$

Remarks :
- The standard Syracuse conjecture corresponds to the particular case $b_1 = 1$. In this case, $B_1 = 0$ and since we have tested the sequence up to $2^{68} > 2^{17}$, there is only one cycle. As a reminder, the conjecture is : "For any initial value, the sequence reaches the value 1."
- Trivial cycles for v : We will call "trivial cycle", any cycle obtained by testing the sequence for $v_0 \leq 2^{N_0+B_1}$
- Very trivial cycles for v : Among the trivial cycles, these cases stand out particularly and will often be omitted
  - If $b_1 > 0$ then we have a cycle for $v_0 = b_1$ for the transition list "10" since $v_1 = \frac{3v_0 + b_1}{2} = \frac{3b_1 + b_1}{2} = 2b_1$ and $v_2 = \frac{v_1}{2} = b_1 = v_0$
  - If $b_1 < 0$ then we have a cycle for $v_0 = -b_1$ for the transition list "1" since $v_1 = \frac{3v_0 + b_1}{2} = \frac{-3b_1 + b_1}{2} = -b_1 = v_0$

  For all value of b_1, there is at least one cycle.
- We will say that we have "tested the sequence" or that the "sequence is verified" up to a value M if for all value less than M, the results are not in disagreement with the theorem, that is, the sequence u does not

diverge and the cycles found are included in the list of trivial cycles. If $M \geq 2^{N_0+B_1}$, then it means that we have found all the trivial cycles and no others.

- We have taken $N_0 = 17$ for the proof.
  We could certainly have taken a lower value , especially for certain values of $-1024 \leq b_1 \leq 1024$
  We could also have taken a higher value like $N_0 = 22$ (or a little more), which would have facilitated the conclusion regarding the search for cycles, and which would not have changed the results of the tests, which would simply have taken a bit longer to perform.
- There is no conceptual difficulty in proving the theorem for $b_1 < -1024$ or $b_1 > 1024$, it would just require increasing the value of N_0 (whose new value could be determined and would likely not be much higher) and again testing the sequence for $v_0 \leq 2^{N_0+B_1}$... until it becomes impossible to perform such a verification, already for $|b_1| > 2^{68-17} = 2^{51}$, which represents the maximum number of tests performed to date for the standard sequence.

:

- We will look for trivial cycles by testing the sequence
- We will construct the JGL list, which is the central element for the proof
- We will finalize the study of cycles by proving that there are no cycles other than trivial cycles.
- We will then prove that the sequence does not diverge.

2. **Extension $5n+1$**

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \dfrac{5v_n+1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

Theorem $5n+1$: There exists at least one value v_0 for which v diverges to infinity

III. Test of Syracuse sequence to find trivial cycles

For the standard Syracuse sequence, Syracuse conjecture has been verified for all $v_0 \leq 1.25 \times 2^{62}$ (source Wikipedia on 02/01/2021)
After checking the Wikipedia[1][2] page at the time of writing this version, (writing began on 10/30/2022), the Syracuse conjecture has been verified for all $v_0 \leq 2^{68}$
This verification bound may still change, which would alter the minimum length of a potential cycle.

We will set $maxSyr\_1 = 2^{68}$ for the rest of the document.

As $2^{68} > 2^{17}$, we have no additional test to perform, the only trivial cycle is 1,4,2 for u or 1,2 for v.

I have not found any verification data for the sequence for $b_1 \neq 1$, it is true that calculations would need to be done for each value of b_1

We will set $B_1 = \lceil \frac{\text{Ln}(|b_1|)}{\text{Ln}2} \rceil$

- Trivial cycles for v : We will call a "trivial cycle", any cycle obtained by testing the sequence for $v_0 \leq 2^{N_0+B_1}$
- Very trivial cycles for v: Among the trivial cycles, these cases stand out particularly and will often be omitted
  - If $b_1 > 0$ then we have a cycle for $v_0 = b_1$ for the transition list "10" since $v_1 = \frac{3v_0+b_1}{2} = \frac{3b_1+b_1}{2} = 2b_1$ and $v_2 = \frac{v_1}{2} = b_1 = v_0$
  - If $b_1 < 0$ then we have a cycle for $v_0 = \text{-}b_1$ for the transition list "1" since $v_1 = \frac{3v_0+b_1}{2} = \frac{\text{-}3b_1+b_1}{2} = \text{-}b_1 = v_0$

  For all value of b_1, there exists at least one cycle.
- For $b_1 < 0$, it is sufficient to study v for $v_0 > \text{-}b_1$.

  Indeed :
  - If $v_0 = \text{-}b_1$ then we have the trivial cycle
  - If $0 < v_0 < \text{-}b_1$, then there exist n such that $v_n < 0$. This thus reduces to the study of v' with $v'_0 = \text{-}v_n > 0$ and $b_1{}' = \text{-}b_1 > 0$.

    Proof of the existence of n such that $v_n < 0$:

    - If v_0 is even, then $v_1 - v_0 = \frac{\text{-}v_0}{2} < 0$ because $v_0 \geq 2$
    - If v_0 is odd, then $v_1 - v_0 = \frac{3v_0+b_1}{2} - v_0 = \frac{v_0+b_1}{2} < 0$

    Therefore the sequence v is strictly decreasing as long as the current term is positive. Since $v_n \neq 0$, there will exist n such that $v_n < 0$
- For $b_1 < 0$, we also have two types of fairly recurrent cycles :
  - For $b_1 = \text{-}3(2k+1)$ with $k > 0$: if $v_0 = \frac{\text{-}b_1}{3} = \text{-}(2k+1)$ which is odd, then $v_1 = \frac{3v_0+b_1}{2} = 0$ and then $v_n = 0$ for all $n > 0$
  - For $b_1 = \text{-}9(2k+1)$ with $k > 0$: if $v_0 = \frac{\text{-}b_1}{9} = 2k+1$ which is odd, then $v_1 = \frac{3v_0+b_1}{2} = \frac{\frac{\text{-}b_1}{3}+b_1}{2} = \frac{b_1}{3}$ is odd, $v_2 = \frac{3v_1+b_1}{2} = \frac{b_1+b_1}{2} = b_1$ is odd and $v_3 = \frac{3v_2+b_1}{2} = \frac{3b_1+b_1}{2} = 2b_1$ is even, $v_4 = \frac{2b_1}{2} = b_1 = v_2$ therefore we have a cycle of length 2, for which the minimum value is b_1

Study of the behavior of v for the values of $v_0 < 2^{N_0+B_1}$, in order to find the set of trivial cycles, the goal of the demonstration being to prove that no others exist :
- The calculations were done with the JavaScript algorithm below. It can certainly be optimized like all the other algorithms provided here, this is not the purpose of this document. This program was used to obtain the results below.
- For $\text{-}1024 < b_1 < 1024$ by testing for $v_0 < 2^{N_0+B_1}$ with $N_0 = 17$:
  All trivial cycles obtained have a length less than or equal to $426 < 1539$ (and $426 < 1054$ for $b_1 < 0$).
  The first condition is satisfied for the demonstration.
  The computation time was 17250 s for $0 < b_1 < 1024$ and 17995 s for $\text{-}1024 < b_1 < 0$, for a total of approximately 10 hours on my computer (Intel(R) Core(TM) i5-1035G1 processor, Google Chrome Version 119.0.6045.160 (Official build) (64 bits)).

$maxSyr\_b1 = 2^{N_0+B_1}$ with $N_0 = 17$.

Value of b_1 : Test for $v_0 < 2^{N_0+B_1}$ with N_0 : 17 Try See/Hide code

Tests for all values of b_1 with 0 $< |b_1| <$ 8 Tests for all values of b_1

Summary of exhaustive results for $0 < b_1 < 1024$:

- 259 values of b_1 have only one very trivial cycle
- The maximum length of a cycle is 426 obtained for $b_1 = 563$ and $v_0 = 19$
- There are 2864 cycles that are not very trivial

Exhaustive results for : $0 < b_1 < 1024$: See/Hide results

Results at the end of the document

See the results

Summary of exhaustive results for $-1024 < b_1 < 0$:

- 256 values of $b1$ have only one very trivial cycle
- The maximum length of a cycle is 426 obtained for $b_1 = -563$ and $v_0 = -19$
- There are 3927 cycles that are not very trivial

Exhaustive results for : $-1024 < b_1 < 0$: See/Hide results

Results at the end of the document

See the results

IV. List of transitions JGL and minimum length of a cycle for v , for all $b_1 > 0$ and $v_0 \in \mathbb{Z}$

1. Purpose

The objective of this section is to rediscover, in a somewhat different way, the result obtained by Mr. Shalom Eliahou[3][4].

- In French: Published on 2011/12/20 in the document "Le problème 3n+1 : y-a-t-il des cycles non triviaux ?" (https://images.math.cnrs.fr/Le-probleme-3n-1-y-a-t-il-des-cycles-non-triviaux-III.html).
- In English : "The 3x + 1 problem : new lower bounds on nontrivial cycle lengths" (https://www.sciencedirect.com/science/article/pii/0012365X9390052U)

These documents are referenced in the Wikipedia pages of the Syracuse conjecture in French[1] and Collatz conjecture[2] in English, even though the result has since been updated.

Since the conjecture in the standard case has been verified for $n \leq 2^{68}$, then there is no other cycle of length less than 186 billion for u.

We will prove that for $0 < b_1 < 1024$, given the tests for $v_0 < 2^{N_0+B_1}$, there indeed exist so-called trivial cycles and the minimum length of another cycle is greater than or equal to 2510 for u and 1539 for v
And, we will prove that for $-1024 < b_1 < 0$, given the tests for $v_0 < 2^{N_0+B_1}$, there indeed exist so-called trivial cycles and the minimum length of another cycle is greater than or equal to 1719 for u and 1054 pour v

Notes :

- When Syracuse conjecture was stated, it was certainly natural to see how far it was true.
- The result of Shalom Eliahou which shows the link between the verification of the conjecture and the minimal length of a cycle, has certainly also contributed to testing the validity of the conjecture even further.
- Computers have become more powerful over the years and certainly $364249198012174112 \sim 3.64 \times 10^{17}$ tests must have been conducted to verify the conjecture up to 2^{68}
- **In fact, in my proof of the theorem, it is sufficient to verify it up to 2^{17} and perhaps one could do less.**

The method used is based on the use of a transition list that we will call JGL (for "Just Greater List") that we will define, which will allow us to construct it step by step.

By a simple computer calculation, which lasts quite a long time, we find the result.

A number of paragraphs are "duplicated" for the case $v_0 < 0$ (or $b_1 < 0$)

In the following section, we will not be satisfied with this calculation and we will add a more mathematical and rigorous proof by studying the patterns of the JGL list to obtain the result instantly, without any real computer calculation.

2. **Approximated reduced Syracuse sequence : v'**

Let's define the approximated sequence by replacing the term $3v_n + b_1$ by $3v_n$
It is immediate that the approximation makes sense for values of v_0 that are large enough and values of n that are low enough.

We ensure that the transitions from v' are identical to those of v

$$\begin{cases} v'_0 = v_0 > 0 \\ v'_{n+1} = \dfrac{v'_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v'_{n+1} = \dfrac{3v'_n}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

We can notice that some elements of v' are not integers.

3. **Property IV-3 : It exists r_n such that $v_n = v'_n + r_n$ for all $n \geq 0$ with $r_n \in \mathbb{Q}$**

This expression is true for $n = 0$ with $r_0 = 0$
Assume that the property is true at rank n and prove that it is verified at the rank $(n+1)$

If v_n is even then $v_{n+1} = \dfrac{v_n}{2} = \dfrac{v'_n + r_n}{2} = \dfrac{v'_n}{2} + \dfrac{r_n}{2} = v'_{n+1} + \dfrac{r_n}{2}$ then it's true with $r_{n+1} = \dfrac{r_n}{2}$

If v_n is odd then $v_{n+1} = \dfrac{3v_n + b_1}{2} = \dfrac{3(v'_n + r_n) + b_1}{2} = \dfrac{3v'_n}{2} + \dfrac{3r_n + b_1}{2}$ therefore it's true with $r_{n+1} = \dfrac{3r_n + b_1}{2}$

It will be noted that if v_0 follows the trajectory L, then r_n follows the same evolution as the reduced Syracuse sequence. It's as if we started from the value 0.

A advantage for the study is that the quantity r_n only depends on the list of transitions (but not on v_0)

Therefore for all n

- If $b_1 > 0$ then $r_n \geq 0$
- If $b_1 < 0$ then $r_n \leq 0$

See paragraph IV-10 for more details

4. **Condition for the existence of a non-trivial cycle , for all $b_1 > 0$ and $v_0 > 0$**

Assume it exists $L(N, m, d)$ which represents a cycle of length N for v.

<u>Note :</u> We always use the reduced sequence v. Thus, if v has a cycle of length N, then u has a cycle of length $N+m$ (and vice versa).

If it exists $L(N, m, d)$ for which we have a cycle for v, then the N circular permutations of $L(N, m, d)$ represent this cycle of length N. We can always choose the transition list for which v_0 is the smallest element of the cycle, it is not a restriction.

If $v_0 > 0$ is the smallest element of a potential cycle of length N then the N distinct elements are $v_0, v_1, \ldots, v_{N-1}$ and $v_{n+1} = v(v_n)$ for all $0 \le n < N$ and v_0 follows the list of transitions $L(N, m, d)$

- $v_n > v_0$ for $0 < n < N$
- $v_N = v_0$ to have a cycle of length N

In this paragraph, we also study the general case with $b_1 > 1$

Moreover, for all $0 < n \le N$,

- $r_n \ge 0$ because $b_1 > 0$ and as t_0 is necessarily of "type 1" so that v_1 is greater than v_0, then we even have $r_n > 0$
- $v_n = \frac{3^{m_n}}{2^n} v_0 + r_n \ge v_0 \Leftrightarrow \left(1 - \frac{3^{m_n}}{2^n}\right) v_0 \le r_n$

  0. **Case 0** : If $\frac{3^{m_n}}{2^n} > 1$, then it is always verified because $v_0 > 0$ and $r_n \ge 0$
  1. **Case 1** : If $\frac{3^{m_n}}{2^n} < 1$, then $\left(1 - \frac{3^{m_n}}{2^n}\right) v_0 \le r_n \Leftrightarrow v_0 \le \frac{r_n}{1 - \frac{3^{m_n}}{2^n}} = V(n-1)$.

Remarks :

- If, for all $0 < n \le N$, we were in the "case 0", then $v_N > v_0$ and there would be no cycle.

  Therefore, necessarily, there exists at least one step n for which $\frac{3^{m_n}}{2^n} < 1$, that is to say that r_n is sufficiently large to compensate for the difference $v'_n - v_0$, the crossing below the boundary induced by v_0.
- The sequence test has been done for $v_0 < maxSyr\_b1$, which made it possible to detect all the "trivial" cycles.

  For $b_1 = 1$, $maxSyr\_b1 = 2^{68}$ and we found only one cycle of length 2 for the transition list "10"

  For $b_1 > 1$, $maxSyr\_b1 = 2^{17+B_1}$, and for each value of b_1, a set of "trivial" cycles was possibly found in addition to the length 2 cycle for the transition list "10"
- For a "non-trivial" cycle, we necessarily have $v_0 > maxSyr\_b1$ and therefore for all the values of $0 < n < N$ corresponding to the "case 1", we have $V(n-1) \ge maxSyr\_b1$

  To increase the value of $V(n-1)$ with v_0 the minimum value of the cycle :
  - r_n must be maximum
  - $1 - \frac{3^{m_n}}{2^n}$ must be the closest to 0 by a higher value

  Necessary condition IV-4-1 : $\frac{r_N}{1 - \frac{3^m}{2^N}} \ge maxSyr\_b1$ to have a non-trivial cycle of length N, with v_0 the minimum value of the cycle

  Necessary and sufficient condition IV-4-2 : $v_0 = \frac{r_N}{1 - \frac{3^m}{2^N}}$ to have a non-trivial cycle of length N, with v_0 the minimum value of the cycle

The JGL list that we will define will satisfy the 2 properties :

- $\frac{3^m}{2^N}$ will be as close as possible to 1 by defect
- r_N will be the maximum for the set $L(N, m, d)$ with v_0 the minimum of v_n for $0 < n < N$

Let us first consider when $\frac{3^m}{2^N}$ is as close as possible to 1 by defect. To achieve this, we need a calculation precision greater than the defect in JavaScript, this is the subject of the following paragraph , after the study of the symmetric case for $b_1 < 0$.

Then we will analyze the behavior of r_N and construct the JGL list.

5. Condition for the existence of a non-trivial cycle , for all $b_1 > 0$ and $v_0 < 0$

For $b_1 < 0$ (initially) :

- We use the equivalence described in the paragraph I-a, namely that the extension for $b_1 < 0$ can be replaced by the study of the sequence $b_1' = -b_1 > 0$ and $v_0 < 0$ since all the elements are simply opposite.
- Therefore, after this change b_1 is always positive and, consequently r_n is also but $v_0 < 0$.

Assume it exists $\mathrm{L}(\mathrm{N}, m, d)$ which represents a cycle of length N for v, it is then a cycle for $-v$.

Note : We always use the reduced sequence v. Thus, if v has a cycle of length N, then u has a cycle of length $\mathrm{N}+m$ (and vice versa).

If it exists $\mathrm{L}(\mathrm{N}, m, d)$ for which we have a cycle for v, then the N circular permutations of $\mathrm{L}(\mathrm{N}, m, d)$ represent this cycle of length N. We can always choose the transition list for which v_0 is the largest element of the cycle, it is not a restriction.

If v_0 is the largest element of a potential cycle of length N then the N distinct elements are $v_0, v_1, \ldots, v_{N-1}$ and $v_{n+1} = v(v_n)$ for all $0 \le n < \mathrm{N}$ and v_0 follows the transition list $\mathrm{L}(\mathrm{N}, m, d)$

NB : Keep attention, here v_0 represents the largest element of the cycle :

- If $v_0 > 0$ then we would be in the previous case which has already been described
- If $v_0 < 0$ then $v_0 \ge v_n$ means $|v_0| \le |v_n|$, this is the symmetrical case that we study here.

We are therefore in the case :

- $v_0 < 0$
- $v_n < v_0$ for $0 < n < \mathrm{N}$
- $v_N = v_0$ to have a cycle of length N

In this paragraph, we also study the general case with $b_1 > 0$ and $v_0 < 0$

Moreover, for all $0 < n \le \mathrm{N}$,

- $r_n \ge 0$ because $b_1 > 0$ after transformation and as t_0 is necessarily of "type 1" so that v_1 is less than v_0, therefore we even have $r_n > 0$
- $v_n = \dfrac{3^{m_n}}{2^n} v_0 + r_n \le v_0 \Leftrightarrow \left(1 - \dfrac{3^{m_n}}{2^n}\right) v_0 \ge r_n$

    0. **Case 0** : If $\dfrac{3^{m_n}}{2^n} < 1$, then it's not possible

    1. **Case 1** : If $\dfrac{3^{m_n}}{2^n} > 1$, then $\left(1 - \dfrac{3^{m_n}}{2^n}\right) v_0 \ge r_n \Leftrightarrow v_0 \le \dfrac{r_n}{1 - \dfrac{3^{m_n}}{2^n}} = \mathrm{V}(n-1)$.

Remarks :

- If, for all $0 < n \le \mathrm{N}$, we were in the "case 0", then there would be no cycle.

  Therefore, necessarily, there is at least one step n for which $\dfrac{3^{m_n}}{2^n} > 1$, that is to say that r_n is sufficiently large to compensate for the difference $v'_n - v_0$, the state under the boundary induced by v_0.

- The sequence test has been done for $-maxSyr\_b1 < v_0 < 0$, which made it possible to detect all the "trivial" cycles (i.e., with $0 < v_0 < maxSyr\_b1$ and $b_1 < 0$).

  $maxSyr\_b1 = 2^{N_0+B_1}$ and for each value of b_1, we may have found a set of "trivial" cycles, in addition to the cycle of length 1 for the transition list "1"
- For a "non-trivial" cycle, we necessarily have $v_0 < -maxSyr\_b1$ and therefore for all the values of $0 < n < \mathrm{N}$ corresponding to "case 1", we have $0 > \mathrm{V}(n-1) \geq -maxSyr\_b1$

  To increase the value of $\mathrm{V}(n-1)$ with v_0 the maximum value of the cycle :
  - r_n must be maximum
  - $1-\dfrac{3^{m_n}}{2^n}$ must be the closest to 0 by lower value

  Necessary condition IV-5-1 : $\dfrac{r_\mathrm{N}}{1-\dfrac{3^m}{2^\mathrm{N}}} \geq -maxSyr\_b1$ to have a non-trivial cycle of length N, with v_0 the maximum value of the cycle

  Necessary and sufficient condition IV-5-2 : $v_0 = \dfrac{r_\mathrm{N}}{1-\dfrac{3^m}{2^\mathrm{N}}}$ to have a non-trivial cycle of length N, with v_0 the maximum value of the cycle

The JGL list that we will define will satisfy the 2 properties :

- $\dfrac{3^m}{2^\mathrm{N}}$ will be as close as possible to 1 by excess
- r_N will be the maximum for the set $\mathrm{L}(\mathrm{N}, m, d)$ with v_0 the maximum of v_n for $0 < n < \mathrm{N}$

Let us first consider when $\dfrac{3^m}{2^\mathrm{N}}$ is as close as possible to 1 by excess. To achieve this, we need a calculation precision greater than the default in JavaScript, this is the subject of the following paragraph.

Then we will analyze the behavior of r_N and construct the JGL list.

The JGL list will be identical to the previous case ($b_1 > 0$ and $v_0 > 0$), only interesting values of N will be different.

6. **Algorithmic note : Use of BigInt() for the decimal computations**

The implementation of decimals in JavaScript language (IEEE 754 standard) does not provide sufficient precision in certain calculations.

Instead of programming or using a library to improve calculation precision (which slows down calculations), since only a few mathematical operations are needed in the programs of the document, mainly division and the natural logarithm, we multiply the quantities by a factor depending on the desired precision. This way, we can use the integer division of BigInt(). We do the same for the natural logarithms of 2 and 3.

We see that the natural logarithm of 5 is also defined, used to study the form $5n+1$.

We then have the following code and global variables for all algorithms (definition of precision, evaluation and formatting of the results) :

```
Javascript code not available in this document.
```

7. **Equivalence of values of (N, m, d) such that $\left|1-\dfrac{3^m}{2^\mathrm{N}}\right| \to 0$ and approximations of X by $\dfrac{m}{d}$**

We have : $\dfrac{m}{d} - X = \dfrac{m}{d} - \dfrac{\mathrm{Ln}2}{\mathrm{Ln}3-\mathrm{Ln}2} = \dfrac{m(\mathrm{Ln}3-\mathrm{Ln}2)-d\,\mathrm{Ln}2}{d(\mathrm{Ln}3-\mathrm{Ln}2)} = \dfrac{m\,\mathrm{Ln}3-(m+d)\mathrm{Ln}2}{d(\mathrm{Ln}3-\mathrm{Ln}2)} = \dfrac{\mathrm{Ln}\left(\dfrac{3^m}{2^N}\right)}{d(\mathrm{Ln}3-\mathrm{Ln}2)}$

because $\mathrm{N} = m+d$

Therefore $\frac{3^m}{2^{\mathrm{N}}} = \mathrm{e}^{d(\mathrm{Ln3}-\mathrm{Ln2})(m/d-X)}$ is close to 1 if and only if $d(\mathrm{Ln3}-\mathrm{Ln2})(\frac{m}{d}-X)$ is close to 0.

It is therefore necessary to look for the approximations of X with $\frac{m}{d}$.

Then, we will have an equivalent because $\mathrm{e}^u \approx 1+u$ near 0, because for approximations, the difference $\left|\frac{m}{d}-X\right|$ is of the order of $\frac{1}{d^2}$

$\frac{3^m}{2^{\mathrm{N}}} \approx 1+d(\mathrm{Ln3}-\mathrm{Ln2})(\frac{m}{d}-X)$

Therefore :

- **Property IV-7-1 : $1-\frac{3^m}{2^{\mathrm{N}}} \approx d(\mathrm{Ln3}-\mathrm{Ln2})(X-\frac{m}{d})$ for approximations of X by $\frac{m}{d}$**

NB : In the rest of the document, we will sometimes note N as an "approximation" of X as a language shortcut, of course, it is $\frac{m}{d}$ which approximates X with $\mathrm{N} = m+d$

We could look for the approximations of X using continuous fractions, as the method allows for rapid convergence

However, this is not the method considered to obtain the exhaustive list of approximations for a given maximum value of N.

The Stern-Brocot tree will be used to list all the defect (or excess) approximations of X.

The method gives all the irreducible fractions $\frac{m}{d}$, which is not necessarily a constraint for the length of a cycle $\mathrm{N} = m+d$ so we will have to to consider all the multiple lengths $k\mathrm{N}$ with $k \geq 1$, if that has a significance for the Syracuse theorem.

8. **Defect approximations of X by $\frac{m}{d}$**

Reminder of the method of the Stern-Brocot tree :

Initially, $a = d = 0$ and $b = \mathrm{c} = 1$, $\frac{1}{0}$ representing infinity.

We repeat the following steps until the desired precision is readched, knowing that we do not necessarily have a better approximation by defect (different as those already obtained) at each step

If $\frac{a}{b} < X < \frac{\mathrm{c}}{d}$, then we compare with the median $\frac{a+\mathrm{c}}{b+d}$.

- If $X \geq \frac{a+\mathrm{c}}{b+d}$ then $\frac{a+\mathrm{c}}{b+d}$ is a better defect approximation and we change $\frac{a}{b}$ by the median
- If $X < \frac{a+\mathrm{c}}{b+d}$ then we change $\frac{\mathrm{c}}{d}$ by the median

We will note **appX(k) = $\frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ the k$^{\text{th}}$ approximation of X by defect** (with $\frac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}} < 1$ and $\mathrm{NX}_k = \mathrm{mX}_k+\mathrm{dX}_k$)

We find the following results :

m	d	N	$diff = \frac{m}{d} - X$	$d^2 \times \lvert diff \rvert$
1	1	2	-7.0951×10^{-1}	7.0951×10^{-1}
3	2	5	-2.0951×10^{-1}	8.3805×10^{-1}
5	3	8	-4.2845×10^{-2}	3.8560×10^{-1}
17	10	27	-9.5113×10^{-3}	9.5113×10^{-1}
29	17	46	-3.6289×10^{-3}	1.0488
41	24	65	-1.1780×10^{-3}	6.7850×10^{-1}
94	55	149	-4.2038×10^{-4}	1.2717
147	86	233	-2.0897×10^{-4}	1.5455
200	117	317	-1.0958×10^{-4}	1.5001
253	148	401	-5.1832×10^{-5}	1.1353
306	179	485	-1.4085×10^{-5}	4.5129×10^{-1}
971	568	1539	-4.2491×10^{-6}	1.3709
1636	957	2593	-2.4094×10^{-6}	2.2067
2301	1346	3647	-1.6331×10^{-6}	2.9587
2966	1735	4701	-1.2049×10^{-6}	3.627
3631	2124	5755	-9.3354×10^{-7}	4.2115
4296	2513	6809	-7.4619×10^{-7}	4.7123
4961	2902	7863	-6.0906×10^{-7}	5.1293
5626	3291	8917	-5.0436×10^{-7}	5.4625
6291	3680	9971	-4.2179×10^{-7}	5.712
6956	4069	11025	-3.5500×10^{-7}	5.8777
7621	4458	12079	-2.9988×10^{-7}	5.9597
8286	4847	13133	-2.5360×10^{-7}	5.9578
8951	5236	14187	-2.1419×10^{-7}	5.8723
9616	5625	15241	-1.8024×10^{-7}	5.7029
10281	6014	16295	-1.5068×10^{-7}	5.4498
10946	6403	17349	-1.2471×10^{-7}	5.1129
11611	6792	18403	-1.0172×10^{-7}	4.6923
12276	7181	19457	-8.1214×10^{-8}	4.1879
12941	7570	20511	-6.2818×10^{-8}	3.5998
13606	7959	21565	-4.6220×10^{-8}	2.9278
14271	8348	22619	-3.1169×10^{-8}	2.1722
14936	8737	23673	-1.7459×10^{-8}	1.3327
15601	9126	24727	-4.9171×10^{-9}	4.0951×10^{-1}
47468	27767	75235	-9.7079×10^{-10}	7.4848×10^{-1}
79335	46408	125743	-1.9476×10^{-10}	4.1945×10^{-1}
190537	111457	301994	-1.4274×10^{-12}	1.7732×10^{-2}
10781274	6306641	17087915	-4.7737×10^{-15}	1.8987×10^{-1}
64497107	37728389	102225496	-5.7097×10^{-16}	8.1273×10^{-1}
118212940	69150137	187363077	-1.8767×10^{-16}	8.9738×10^{-1}

171928773	100571885	272500658	-4.3877×10^{-17}	4.4380×10^{-1}
397573379	232565518	630138897	-1.1224×10^{-18}	6.0709×10^{-2}
6586818670	3853041921	10439860591	-6.4797×10^{-21}	9.6198×10^{-2}
72057431991	42150895613	114208327604	-3.2245×10^{-22}	5.7290×10^{-1}
137528045312	80448749305	217976794617	-2.7550×10^{-23}	1.7830×10^{-1}
890638885193	520990349522	1411629234715	-3.6910×10^{-24}	1.0018
1643749725074	961531949739	2605281674813	-1.6948×10^{-24}	1.5669
2396860564955	1402073549956	3798934114911	-9.5300×10^{-25}	1.8734
3149971404836	1842615150173	4992586555009	-5.6592×10^{-25}	1.9214
3903082244717	2283156750390	6186238995107	-3.2822×10^{-25}	1.711
4656193084598	2723698350607	7379891435205	-1.6742×10^{-25}	1.242
5409303924479	3164239950824	8573543875303	-5.1387×10^{-26}	5.1451×10^{-1}

It should be noted that for N = 301994, the approximation is excellent in precision ($\left|\frac{m}{d}-X\right| < \frac{1}{2d^2} = \frac{1.77\times10^{-2}}{d^2}$), it is indeed a better approximation, a fraction obtained from the continued fraction expansion of X.

Test of defect approximation of X with fractions : Try See/Hide code Back to approximations by excess

9. **Approximations by excess of X by $\frac{m}{d}$**

This will be used for the proof of the experimental results in the following section
but also in the case $b_1 > 0$ with $v_0 < 0$
We can easily find the successive approximations (by fractions) by excess of X with the principle of the Stern-Brocot tree.

Reminder of the method :

Initially, $a = d = 0$ and $b = c = 1$, $\frac{1}{0}$ representing infinity.

We repeat the following steps until the desired precision is reached, knowing that we do not necessarily have a better approximation by excess (different as those already obtained) at each step

If $\frac{a}{b} < X < \frac{c}{d}$, then we compare with the median $\frac{a+c}{b+d}$.

- If $X \leq \frac{a+c}{b+d}$ then $\frac{a+c}{b+d}$ is a better excess approximation and we change $\frac{c}{d}$ by the median
- If $X > \frac{a+c}{b+d}$ then we change $\frac{a}{b}$ by the median

We find the following results :

m	d	N	$diff = \frac{m}{d} - X$	$d^2\times\lvert diff\rvert$
2	1	3	2.9049×10^{-1}	2.9049×10^{-1}
7	4	11	4.0489×10^{-2}	6.4782×10^{-1}
12	7	19	4.7744×10^{-3}	2.3395×10^{-1}

53	31	84	1.6613×10^{-4}	1.5965×10^{-1}
359	210	569	1.2518×10^{-5}	5.5205×10^{-1}
665	389	1054	2.7677×10^{-7}	4.1881×10^{-2}
16266	9515	25781	6.5991×10^{-9}	5.9746×10^{-1}
31867	18641	50508	9.6119×10^{-10}	3.3400×10^{-1}
111202	65049	176251	1.3650×10^{-10}	5.7758×10^{-1}
301739	176506	478245	4.9404×10^{-11}	1.5392
492276	287963	780239	2.9730×10^{-11}	2.4653
682813	399420	1082233	2.1035×10^{-11}	3.3559

Test of excess approximation of X with fractions : Try See/Hide code (it's the same code as for the defect approximations)

10. **Expanded expression of r_n**

What is essential to remember is that for a list of transitions of length N :

- r_n is of the sign of b_1
- The more the transitions of "type 1" are at the end of the list of transitions, the larger r_N is
- The more the transitions of "type 0" are at the beginning of the list of transitions, the larger r_N is
- Swapping two successive transitions does not have a significant impact on the value of r_N

We simply state a formula that we will prove by induction, after looking at an example to illustrate the idea of the formula.

- if $m_n = 0$ then $r_n = 0$
- if $m_n > 0$ then $r_n = \frac{b_1}{2^n} \sum_{i=1}^{m_n} 3^{m_n - i} \times 2^{\text{ind(i)}} = \frac{3^{m_n} b_1}{2^n} \sum_{i=1}^{m_n} \frac{2^{\text{ind(i)}}}{3^i}$

with

- m_n is the number of "type 1" transitions among the n first transitions of L
- ind(i) is the index (starting from 0) of the i$^{\text{th}}$ transition (starting from 1) of "type 1" in the list L

The interest of this formula is to see :

- that for $b_1 \neq 1$, the value of r_n is the one obtained for $b_1 = 1$ multiplied by b_1
- that for fixed m_n, r_n will be greater when the values ind(i) will be larger, i.e. when the "type 1" transitions will be closer to the end or what is equivalent, when the "type 0" transitions will be at the top of the list (at least as much as possible given the constraints imposed on L).
- that the permutation of two transitions (*10* into *01*), for the i$^{\text{th}}$ transition of type "1", generates a modification of r_n equal to $\frac{3^{m_n} b_1}{2^n} \times \frac{2^{\text{ind(i)}}}{3^i}$ because ind(i) is increased by 1 and the other terms are unchanged.

  The relative difference is therefore $\dfrac{\frac{2^{\text{ind(i)}}}{3^i}}{\sum_{i=1}^{m_n} \frac{2^{\text{ind(i)}}}{3^i}}$
- that if two transition lists having the same length N and the same number m_N of transitions of type "1" (thus same factor $\frac{3^{m_N} b_1}{2^N}$) and that if in addition
  - they differ only from p permutations (all in the same direction to maximize the difference)
  - that for each "type 1" transition of each of the lists, we have $1 < \frac{3^i}{2^{\text{ind(i)}}} < 2$ therefore $\frac{1}{2} < \frac{2^{\text{ind(i)}}}{3^i} < 1$

then the relative difference of r_N for both lists is less than $\frac{2p}{m_N}$, which can be very small.

For example, in the case of the standard Syracuse sequence, we consider the list of transitions L = "1101".

- $r_0 = 0$
- $t_0 = 1$ then $r_1 = \frac{1}{2}(3r_0+1) = \frac{1}{2} = \frac{1}{2^1}\times 3^0\times 2^1 = \frac{3^1}{2^1}\times\frac{2^0}{3^1}$
- $t_1 = 1$ then $r_2 = \frac{1}{2}(3r_1+1) = \frac{1}{2^2}(2^1\times 3r_1+2^1) = \frac{1}{2^2}(3\times 1+2^1) = \frac{1}{2^2}(3\times 2^0+2^1) = \frac{3^2}{2^2}\left(\frac{2^0}{3^1}+\frac{2^1}{3^2}\right)$
- $t_2 = 0$ then $r_3 = \frac{1}{2}r_2 = \frac{3^2}{2^3}\left(\frac{2^0}{3^1}+\frac{2^1}{3^2}\right)$
- $t_3 = 1$ then $r_4 = \frac{1}{2}(3r_3+1) = \frac{1}{2^4}(2^3\times 3r_3+2^3) = \frac{1}{2^4}(3(3\times 2^0+2^1)+2^3) = \frac{1}{2^4}(3^2\times 2^0+3^1\times 2^1+3^0\times 2^3) = \frac{3^3}{2^4}\left(\frac{2^0}{3^1}+\frac{2^1}{3^2}+\frac{2^3}{3^3}\right)$

We now prove the formula of r_n by induction.
We verify the formula at rank 1 :

- If L = "0" then $m_1 = 0$ and $r_1 = 0$ and there is no term in the sum
- If L = "1" then $m_1 = 1$ and $r_1 = \frac{b_1}{2} = \frac{b_1}{2}\times 3^0\times 2^0 = \frac{3^1 b_1}{2^1}\times\frac{2^0}{3^1}$, which ends the verification at rank 1.

We assume the formula is true for rank n and show that it is true for rank $n+1$

- $t_n = 0$ then $m_{n+1} = m_n$ and $r_{n+1} = \frac{r_n}{2} = \frac{b_1}{2^{n+1}}\sum_{i=1}^{m_n} 3^{m_n - i}\times 2^{\text{ind(i)}} = \frac{b_1}{2^{n+1}}\sum_{i=1}^{m_{n+1}} 3^{m_{n+1}-i}\times 2^{\text{ind(i)}}$
- $t_n = 1$ then $m_{n+1} = m_n + 1$ and $r_{n+1} = \frac{3r_n+b_1}{2} = \frac{1}{2^{n+1}}(3\times 2^n r_n + 2^n b_1) =$
$\frac{1}{2^{n+1}}\left(\left(3b_1\sum_{i=1}^{m_n} 3^{m_n-i}\times 2^{\text{ind(i)}}\right)+2^n b_1\right) = \frac{b_1}{2^{n+1}}\left(\left(\sum_{i=1}^{m_n} 3^{m_{n+1}-i}\times 2^{\text{ind(i)}}\right)+3^0\times 2^n\right) =$
$\frac{b_1}{2^{n+1}}\sum_{i=1}^{m_{n+1}} 3^{m_{n+1}-i}\times 2^{\text{ind(i)}}$

This ends the proof of the formula.

11. **Construction of the JGL(N) transition list for $v_0 > 0$:**

JGL(N) is made up of N specific transitions t_0, t_1... , t_{N-1}.

JGL(N) satisfies two constraints :

1. For all $0 < n \le N$, $v_n \ge v_0$, that is v_0 is the smallest element or $\frac{v_n}{v_0} \ge 1$
2. Priority is given to transitions of "type 0", therefore, for all $0 < n \le N$, $\frac{v_n}{v_0}$ is minimum

Note : If we have JGL(N) then we also have the value of r_N for this list of transitions.

The JGL(N) transition list is the longest transition list that maximize r_N for a given value of m, with v_0 the minimum of v_n, since the "type 1" transitions are placed as late as possible, which maximizes r_n according to the previous paragraph (cf IV-10).

Steps of construction :

They are the same for $b_1 > 1$, only the bounds that we obtain are to be multiplied by b_1 because r_n is multiplied by b_1

a. **JGL(1) :**

It is a matter of determining the transition t_0

We have $v_0 = \frac{3^0}{2^0} v_0 + r_0$ with $r_0 = 0$

To have $v_1 \geq v_0$, necessarily $t_0 = 1$ (of "type 1") and

$v_1 = \frac{3 v_0 + b_1}{2} = \frac{3^1}{2^1} v_0 + r_1$ with $r_1 = \frac{3 r_0 + b_1}{2} = \frac{b_1}{2}$ and $m_1 = 1, d_1 = 0$

and JGL(1) = "1"

b. **JGL(2) :**

It is a matter of determining the transition t_1

We test whether we can make a "type 0" transition since they are given priority, as we must do them as early as possible.

If $t_1 = 0$ then $v_2 = \frac{v_1}{2} = \frac{3}{4} v_0 + \frac{r_1}{2} = \frac{3}{4} v_0 + \frac{b_1}{4}$

And the condition $v_2 \geq v_0 \Leftrightarrow \frac{3}{4} v_0 + \frac{b_1}{4} \geq v_0 \Leftrightarrow v_0 \leq b_1$

We then test $v_0 = b_1$, then $v_1 = 2 b_1$ and $v_2 = b_1$, it's the trivial cycle, even for $b_1 > 1$, we find it.

Very well, but since we are looking for potential other cycles and the Syracuse sequence is verified for all $v_0 \leq 2^{68}$ (or $v_0 < maxSyr\_b1$ for $b_1 \neq 1$) then necessarily $v_0 > b_1$ and then, necessarily, $t_1 = 1$

Therefore $v_2 = \frac{3 v_1 + b_1}{2} = \frac{3\left(\frac{3}{2} v_0 + \frac{b_1}{2}\right) + b_1}{2} = \frac{9}{4} v_0 + \frac{5 b_1}{4} = \frac{3^2}{2^2} v_0 + \frac{5 b_1}{4}$ or $r_2 = \frac{3 r_1 + b_1}{2} = \frac{5 b_1}{4}$

Finally $m_2 = 2, d_2 = 0$ and

JGL(2) = "11"

c. **JGL(3) :**

It is a matter of determining the transition t_2

As $v_2 > \frac{9}{4} v_0$, then the "type 0" transition ($t_2 = 0$) is possible because $\frac{v_2}{2} \geq \frac{9}{8} v_0 \geq v_0$

Then $v_3 = \frac{3^2}{2^3} v_0 + \frac{5 b_1}{8}$ with $r_3 = \frac{5 b_1}{8}$, $m_3 = 2, d_3 = 1$

and JGL(3) = "110"

d. **JGL(4) :**

It is a matter of determining the transition t_3, by first testing a transition of "type 0"

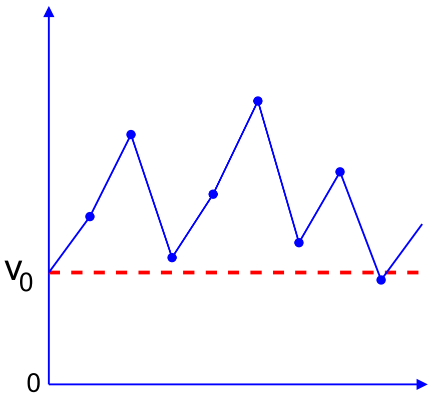
If $t_3 = 0$ then $v_4 = \frac{9}{16} v_0 + \frac{5 b_1}{16}$

And the condition $v_4 \geq v_0 \Leftrightarrow \frac{9}{16} v_0 + \frac{5 b_1}{16} \geq v_0 \Leftrightarrow v_0 \leq \frac{16}{7} \times \frac{5 b_1}{16} = \frac{5 b_1}{7}$, which is in contradiction with

$v_0 \geq maxSyr\_b1$ therefore $t_3 = 1$ and $r_4 = \frac{3 r_3 + b_1}{2} = \frac{23 b_1}{16}$, $m_4 = 3, d_4 = 1$

and JGL(4) = "1101"

e. **General case, construction of JGL($n+1$) from JGL(n) :**



More generally, we assume JGL(n) is constructed therefore $v_n = \frac{3^{m_n}}{2^n} v_0 + r_n \geq v_0$ and we are trying to construct JGL($n+1$), that is to determine t_n, by testing if a "type 0" transition is possible.

Does $v_{n+1} = \frac{v_n}{2} = \frac{3^{m_n}}{2^{n+1}} v_0 + \frac{r_n}{2} \geq v_0 \Leftrightarrow \left(1 - \frac{3^{m_n}}{2^{n+1}}\right) v_0 \leq \frac{r_n}{2}$?

- If $\frac{3^{m_n}}{2^{n+1}} > 1 \Leftrightarrow m_n \mathrm{Ln3} - (n+1)\mathrm{Ln2} > 0$ then the inequality is true for all $n > 0$ because $r_n > 0$ and the natural logarithm function is increasing.

  In this case, $t_n = 0$ and JGL($n+1$) = JGL(n) + "0"
- If $\frac{3^{m_n}}{2^{n+1}} < 1$ then $t_n = 0$ is possible if

  $$\left(1 - \frac{3^{m_n}}{2^{n+1}}\right) v_0 \leq \frac{r_n}{2} \Leftrightarrow \frac{2^{n+1} - 3^{m_n}}{2^n} v_0 \leq r_n \Leftrightarrow v_0 \leq \frac{2^n r_n}{2^{n+1} - 3^{m_n}} = \mathrm{VMax}(n)$$

  If $\mathrm{VMax}(n) \leq maxSyr\_1$ then with $t_n = 0$, we would possibly have a trivial cycle. We eliminate this case and therefore $t_n = 1$, so JGL($n+1$) = JGL(n) + "1".

  The first time $\mathrm{VMax}(n) > maxSyr\_1$ then $t_n = 0$ is possible and therefore JGL($n+1$) = JGL(n) + "0", this is the necessary condition to have a non-trivial cycle of length $n+1$.

  As $r_{n+1} > 0$, to have a cycle of length $n+1$, it is imperative that $v'_{n+1} < v_0$ and r_{n+1} is large enough so that $v_{n+1} = v'_{n+1} + r_{n+1} \geq v_0$, equality would mean a cycle (and $n+1$ would be the minimum length of a non-trivial cycle).

Then, we can test when this inequality is true by simply doing calculations, step by step, it's easy to program.

12. **Construction of the JGL(N) transition list for $v_0 < 0$:**

JGL(N) is made up of N specific transitions t_0, t_1... , t_{N-1}.

JGL(N) satisfies two constraints :

1. For all $0 < n \le N$, $v_n \le v_0$, that is v_0 is the largest element or $\frac{v_n}{v_0} \ge 1$
2. Priority is given to transitions of "type 0", therefore, for all $0 < n \le N$, $\frac{v_n}{v_0}$ is minimum

Note : If we have JGL(N) then we also have the value of r_{N} for this list of transitions.

The JGL(N) transition list is the longest transition list that maximize r_{N} for a given value of m, with v_0 the maximum of v_n, since the "type 1" transitions are placed as late as possible, which maximizes r_n according to the previous paragraph (cf IV-10).

We call the list JGL because it is identical to that described for the case $v_0 > 0$.

Indeed, we can bring together the two cases as follows :

1. For all $0 < n \le N$, $|v_n| \ge |v_0|$, that is, $\frac{v_n}{v_0} \ge 1$
2. Priority is given to "type 0" transitions so for all $0 < n \le N$, $\frac{v_n}{v_0}$ is minimum

It is just considered in different places, namely for the "approximations of X" by defect when $v_0 > 0$ and for the "approximations of X" by excess when $v_0 < 0$

Steps of construction :

They are the same for $b_1 > 1$, only the bounds that we obtain are to be multiplied by b_1 because r_n is multiplied by b_1

a. **JGL(1) :**

It is a matter of determining the transition t_0

We have $v_0 = \frac{3^0}{2^0} v_0 + r_0$ with $r_0 = 0$

To have $v_1 \leq v_0$, necessarily $t_0 = 1$ (of "type 1") and

$$v_1 = \frac{3 v_0 + b_1}{2} = \frac{3^1}{2^1} v_0 + r_1 \text{ with } r_1 = \frac{3 r_0 + b_1}{2} = \frac{b_1}{2} \text{ and } m_1 = 1, d_1 = 0$$

and JGL(1) = "1"

Notes :

- $v_1 = \frac{3 v_0 + b_1}{2} \leq v_0 \Leftrightarrow v_0 \leq -b_1$
- $v_1 = v_0$ for $v_0 = -b_1$, we find the trivial cycle.

As we assume the sequence tested for $-maxSyr\_b1 < v_0 < 0$, we are in the case $v_0 < -maxSyr\_b1 < -b_1$

b. **JGL(2) :**

It is a matter of determining the transition t_1
We test whether we can make a "type 0" transition since they are given priority, as we must do them as early as possible.

If $t_1 = 0$ then $v_2 = \frac{v_1}{2} = \frac{3}{4} v_0 + \frac{r_1}{2} = \frac{3}{4} v_0 + \frac{b_1}{4}$

And the condition $v_2 \leq v_0 \Leftrightarrow \frac{3}{4} v_0 + \frac{b_1}{4} \leq v_0 \Leftrightarrow v_0 \geq b_1$, which is impossible because $v_0 < 0$ and $b_1 > 0$.

Therefore, necessarily, $t_1 = 1$

And $v_2 = \frac{3 v_1 + b_1}{2} = \frac{3\left(\frac{3}{2} v_0 + \frac{b_1}{2}\right) + b_1}{2} = \frac{9}{4} v_0 + \frac{5 b_1}{4} = \frac{3^2}{2^2} v_0 + \frac{5 b_1}{4}$ or $r_2 = \frac{3 r_1 + b_1}{2} = \frac{5 b_1}{4}$

Finally $m_2 = 2$, $d_2 = 0$ and

JGL(2) = "11"

c. **JGL(3) :**

It is a matter of determining the transition t_2, by first testing a transition of "type 0"

If $t_2 = 0$ then $v_3 = \frac{v_2}{2} = \frac{9}{8} v_0 + \frac{5}{8} b_1$

And $v_3 \leq v_0 \Leftrightarrow v_0 \leq -5 b_1$, which is true because we are in the case $v_0 < -maxSyr\_b1$

Then $v_3 = \frac{3^2}{2^3} v_0 + \frac{5 b_1}{8}$ with $r_3 = \frac{5 b_1}{8}$, $m_3 = 2$, $d_3 = 1$

and JGL(3) = "110"

d. **JGL(4) :**

It is a matter of determining the transition t_3, by first testing a transition of "type 0"

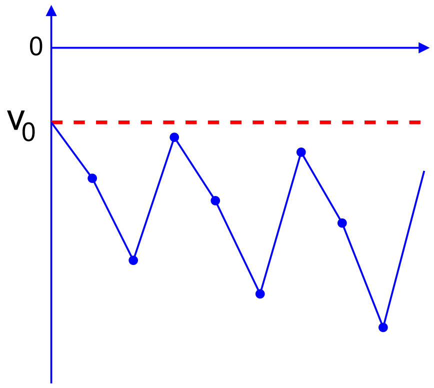
If $t_3 = 0$ then $v_4 = \frac{9}{16} v_0 + \frac{5 b_1}{16}$

And the condition $v_4 \leq v_0 \Leftrightarrow \frac{9}{16} v_0 + \frac{5 b_1}{16} \leq v_0 \Leftrightarrow v_0 \geq \frac{16}{7} \times \frac{5 b_1}{16} = \frac{5 b_1}{7} > 0$, which is in contradiction

with $v_0 < 0$ therefore $t_3 = 1$ and $r_4 = \frac{3r_3 + b_1}{2} = \frac{23b_1}{16}$, $m_4 = 3$, $d_4 = 1$
and JGL(4) = "1101"

We find the same beginning of JGL as for $v_0 > 0$

e. **General case, construction of JGL($n+1$) from JGL(n) :**



More generally, we assume JGL(n) is constructed therefore $v_n = \frac{3^{m_n}}{2^n} v_0 + r_n \leq v_0$ and we are trying to construct JGL($n+1$), that is to determine t_n, by testing if a "type 0" transition is possible.

Does $v_{n+1} = \frac{v_n}{2} = \frac{3^{m_n}}{2^{n+1}} v_0 + \frac{r_n}{2} \leq v_0 \Leftrightarrow \left(1 - \frac{3^{m_n}}{2^{n+1}}\right) v_0 \geq \frac{r_n}{2}$?

- If $\frac{3^{m_n}}{2^{n+1}} < 1 \Leftrightarrow m_n \mathrm{Ln}3 - (n+1)\mathrm{Ln}2 < 0$ then the inequality is always false for all $n > 0$ because $r_n > 0$ and $v_n < 0$ (and the natural logarithm function is increasing for transformation).

  In this case, $t_n = 1$ and JGL($n+1$) = JGL(n) + "1"
- If $\frac{3^{m_n}}{2^{n+1}} > 1$ then $t_n = 0$ is possible if

$$\left(1 - \frac{3^{m_n}}{2^{n+1}}\right) v_0 \geq \frac{r_n}{2} \Leftrightarrow \frac{2^{n+1} - 3^{m_n}}{2^n} v_0 \geq r_n \Leftrightarrow v_0 \leq \frac{2^n r_n}{2^{n+1} - 3^{m_n}} = \mathrm{VMax}(n)$$

  Therefore, it is always possible to have values of v_0 which satisfies this inequality, therefore $t_n = 0$ and therefore JGL($n+1$) = JGL(n) + "0".

  But, as $r_{n+1} = \frac{r_n}{2}$ and $v'_{n+1} < v_0$, to have a cycle of length $n+1$, it is imperative that $r_{n+1} > 0$ is large enough so that $v_{n+1} = v'_{n+1} + r_{n+1} \geq v_0$, equality would mean a cycle.

  Therefore a necessary condition to have a cycle, is $\mathrm{VMax}(n) \leq v_0 < 0$, the previous inequality in the other direction.

  If $\mathrm{VMax}(n) \geq$ -*maxSyr_b1* , then it would be a trivial cycle, which we are not interested in here.

  If $\mathrm{VMax}(n) \leq$ -*maxSyr_b1* then the necessary condition to have a cycle of length $n+1$ is filled (it remains to find a possible v_0 for which the equality would be verified).

remains to find a possible v_0 for which the equality would be verified)

Then, we can test when this inequality is true by simply doing calculations, step by step, it's easy to program.

13. **Minimum length of a cycle obtained through step-by-step calculations for $v_0 > 0$**

In the step-by-step construction of the JGL transition list, we list the values of increasing $\mathrm{VMax}(n)$.

For the first value of $\mathrm{VMax}(n)$ that is greater than *maxSyr_b1* , we get $n+1$, the minimum length of a non-trivial cycle (if it exists).

If $\mathrm{VMax}(n) > maxSyr\_b1$, we change the transition (which corresponds to a simple permutation with the next one) and it is indicated that this length $n+1$ should be studied. If we are trying to construct JGL for values of N much higher than the minimum value of a cycle, this case will repeat itself more and more often and that has no interest.

If we set an very large arbitrary value of *maxSyr_b1* , we will then obtain the list of values of v_0 for which the sequence should be tested to increase the minimum length of a non-trivial cycle.

In the calculation algorithm, we take $b_1 = 1$, but if $b_1 > 1$, then as r_n is multiplied by b_1 compared to the value obtained with $b_1 = 1$, the bound obtained is also to be multiplied by b_1. This also explains the form of $maxSyr\_b1 = 2^{N_0+B_1}$ with $B_1 = \lceil \frac{\mathrm{Ln}(|b_1|)}{\mathrm{Ln}2} \rceil$ which increases the value of v_0 for the line corresponding to $b_1 = 1$

We get the following results , the values of v_0 are to be multiplied by b_1 :

m	d	n	r_n	maximum value for v_0
1	0	1	0.5	$1 \sim 2^{0}$
3	1	4	1.438	$4.6 \sim 2^{2.202}$
5	2	7	2.492	$24.54 \sim 2^{4.617}$
17	9	26	8.172	$108.01 \sim 2^{6.755}$
29	16	45	13.93	$281.94 \sim 2^{8.139}$
41	23	64	19.766	$867.14 \sim 2^{9.760}$
94	54	148	45.156	$2419.68 \sim 2^{11.241}$
147	85	232	70.599	$4862.05 \sim 2^{12.247}$
200	116	316	96.094	$9266.54 \sim 2^{13.178}$
253	147	400	121.643	$19584.90 \sim 2^{14.257}$
306	178	484	147.246	$72058.07 \sim 2^{16.137}$
971	567	1538	466.89	$238670.39 \sim 2^{17.865}$
1636	956	2592	786.548	$420841.76 \sim 2^{18.683}$
2301	1345	3646	1106.22	$620858.37 \sim 2^{19.244}$
2966	1734	4700	1425.906	$841477.43 \sim 2^{19.683}$
3631	2123	5754	1745.606	$1086054.96 \sim 2^{20.051}$
4296	2512	6808	2065.32	$1358717.75 \sim 2^{20.374}$
				...

and after several long hours (or possibly days) :

m	d	n	r_n	max such as $v_0 \leq 2^{\max}$

79335	46407	125742	38152.167	32.277
190537	111456	301993	91629.092	39.369
10781274	6306640	17087914	5184696.98	47.594
64497107	37728388	102225495	31016552.171	50.657
118212940	69150136	187363076	56848407.453	52.262
171928773	100571884	272500657	82680262.824	54.359
397573379	232565517	630138896	191192380.914	59.648
6586818670	3853041920	10439860590	3167590210.299	67.084
72057431991	42150895612	114208327603	34652299930.941	71.413

Test to get the maximum value of v_0 : 2024 knowing that the sequence has been verified up to 2^ 17 Try See/Hide code

We notice that :

- since Syracuse sequence has been verified for all $n \le 2^{68}$, this inequality is never true for values of $n \le 10439860590$, because $2^{67.084} < 2^{68}$ even if there are small truncation errors, due to the numerous calculations. Obviously, if the sequence were verified for larger values of v_0, the value would be even greater for the inequality to be verified.
- $\frac{m_n}{d_n+1}$ are exactly and only those that correspond to approximations by defect of $X = \frac{\text{Ln2}}{\text{Ln3} - \text{Ln2}}$

The JGL(n) transition list fully meets the 2 optimal conditions for having a cycle (cf IV-4-1)

We can therefore affirm, solely based on the results obtained through calculation, in the case where $\frac{3^{m_n}}{2^{n+1}} < 1$,

that :

- For $n \le 10439860590$, t_n is of "type 1" and JGL($n+1$) = JGL(n) + "1"
- For $n > 10439860590$, it may happen that from time to time (perhaps once every 10 billion of transitions), a transition t_n is of "type 0", and then t_{n+1} would be of "type 1" (and if not, it would have been "type 0"), it is a simple interchanging of two transitions.

Conclusion of calculations knowing that the sequence has been verified for 2^{68} :

- **The minimum length for a cycle is 114208327604 for the reduced sequence of Syracuse *v***
- **The minimum length for a cycle is 186265759595 for the standard sequence of Syracuse *u*** because we add m steps

We realize that it would be necessary to verify the sequence for $n \le 2^{72}$ to increase the minimum length of a non-trivial cycle, for which much more calculation time would be needed.
For $b_1 > 1$, the bounds need to be adjusted, and if all the cycles have been verified and searched for $v_0 < 2^{17+B_1}$, then we can affirm that another cycle, if it exists, has a length greater than or equal to $1538 + 1 = 1539$ for v i.e. $1539 + 971 = 2510$ for u

14. Minimum length of a cycle obtained through step-by-step calculations for $v_0 < 0$

In the step-by-step construction of the JGL transition list, we list the values of increasing $|\text{VMax}(n)|$.

For the first value of $|\text{VMax}(n)|$ which is greater than *maxSyr_b1* , we get $n+1$, the minimum length of a non-trivial cycle (if it exists).

If we set an very large arbitrary value of *maxSyr_b1* , we will then obtain the list of values of v_0 for which the sequence should be tested to increase the minimum length of a non-trivial cycle.

In the calculation algorithm, we take $b_1 = -1$ and $v_0 > 0$ (which is equivalent to $b_1 = 1$ and $v_0 < 0$), but if $b_1 < -1$, then as r_n is multiplied by b_1 compared to the value obtained with $b_1 = -1$, the bound obtained is also to be multiplied by $|b_1|$. This also explains the form of $maxSyr\_b1 \;= 2^{N_0+B_1}$ with $B_1 = \lceil \frac{\text{Ln}(|b_1|)}{\text{Ln2}} \rceil$ which increases the value of v_0 for the line corresponding to $b_1 = -1$

We get the following results , the values of v_0 are to be multiplied by $|b_1|$:

m	d	n	r_n	maximum value for v_0
2	0	2	-1.25	$5 \sim 2^{2.322}$
7	3	10	-3.679	$27.10 \sim 2^{4.760}$
12	6	18	-5.984	$219.31 \sim 2^{7.777}$
53	30	83	-25.682	$6143.16 \sim 2^{12.585}$
159	92	251	-77.208	$6143.16 \sim 2^{12.585}$
212	123	335	-103.052	$6143.16 \sim 2^{12.585}$
318	185	503	-154.901	$6143.16 \sim 2^{12.585}$
359	209	568	-172.902	$81063.35 \sim 2^{16.307}$
665	388	1053	-319.971	$3664765.05 \sim 2^{21.805}$
1330	777	2107	-639.956	$3664765.05 \sim 2^{21.805}$
1995	1166	3161	-959.955	$3664765.05 \sim 2^{21.805}$
16266	9514	25780	-7822.559	$153625412.52 \sim 2^{27.195}$
31867	18640	50507	-15325.01	$1054720842.54 \sim 2^{29.974}$
				...

and after several long hours (or possibly days) :

m	d	n	r_n	max such as $v_0 \leq 2^{\max}$
301739	176505	478244	-145106.21	$20520083760.05 \sim 2^{34.256}$
6398923	3743129	10142052	-3077233.924	$1045634943660.33 \sim 2^{39.928}$
32153285	18808465	50961750	-15462461.988	$277238656568662.88 \sim 2^{47.978}$
3406231638	1992517776	5398749414	-1638051164.939	$978964046509643500 \sim 2^{59.764}$
32536519971	19032644086	51569164057	-15646758670.581	$141656386397821760000 \sim 2^{66.941}$
39123338641	22885686007	62009024648	-18814348880.721	$208562895352680150000 \sim 2^{67.499}$
45710157311	26738727928	72448885239	-21981939090.829	$314192776498692500000 \sim 2^{68.090}$
52296975981	30591769849	82888745830	-25149529300.905	$505854005668244600000 \sim 2^{68.777}$
58883794651	34444811770	93328606421	-28317119510.949	$960856011060574000000 \sim 2^{69.703}$
65470613321	38297853691	103768467012	-31484709720.961	$3.41 \times 10^{21} \sim 2^{71.532}$

Test to get the maximum value of v_0 : 2024 knowing that the sequence has been verified up to 2^ 17

We can vary the precision to see more or fewer multiples of the irreducible fractions : 50

Try See/Hide code

We notice that :

- $\frac{m_n}{d_n+1}$ which give a value of $\text{VMax}(n)$ significantly different from the maximum value of v_0 are those and only those that correspond to approximations by excess of $X = \frac{\text{Ln2}}{\text{Ln3} - \text{Ln2}}$.
- The other fractions correspond to non-irreducible fractions, whose numerators and denominators are multiples of the fractions corresponding to the approximations by excess of X

The JGL(n) transition list fully meets the 2 optimal conditions for having a cycle (cf IV-5-1)

For $b_1 > 1$, the bounds need to be adjusted, and if all the cycles have been verified and searched for $v_0 < 2^{17+B_1}$, then we can affirm that another cycle, if it exists, has a length greater than or equal to $1053+1 = 1054$ for v i.e. $1054 + 665 = 1719$ for u
We realize that it would be necessary to verify the sequence for $n \leq 2^{22+B_1}$ to increase the minimum length for a non-trivial cycle to 25781 for v, which may require a very long calculation time.

15. **Step by step JGL transition list**

We notice that in the previous paragraphs of the JGL construction, the cases are presented in a different order but correspond to the same thing.

For $v_0 > 0$, as long as $|\text{VMax}(n)| < maxSyr\_b1$ and for $v_0 < 0$, for all n, we have :

$t_n = 0$ if and only if $\frac{3^{m_n}}{2^{n+1}} > 1 \Leftrightarrow m_n \text{Ln3} - (n+1)\text{Ln2} > 0$ because the natural logarithm function is increasing.

We can also propose a simplified version of the calculation of JGL(n) :
Javascript code not available in this document.

We get JGL(65) = "11011011010110110101101101101011011010110110110101101101011011011"

Test of the short version of JGL(n) : 65 Try

16. **Expression of m_N of JGL(N)**

To go a little further, let's try to find an expression m_N for the boundary transition list JGL
Consider the general case of a sequence defined as follows :
Let $f(n) = a\, m_{n-1} - n\, b$, with $a > 0$ and $b > 0$

- $m_0 = 0$
- $m_n = m_{n-1} + 1$ if $f(n) < 0$
- $m_n = m_{n-1}$ if $f(n) \geq 0$

We realize that the reduced Syracuse sequence corresponds to the particular case $a = \text{Ln3}$ and $b = \text{Ln2}$ according to the previous paragraph. Keep attention, m_n is indexed from 1, i.e. the number of "type 1" transtions of JGL(N) is m_N for $N \geq 1$

The condition $f(n) < 0$ is equivalent to $a\, m_{n-1} < n\, b$ or even $m_{n-1} < \frac{n\, b}{a}$, as $a > 0$

The sequence m_n will increase by 1 at each step as long as $m_{n-1} < \frac{n\, b}{a} \leq \lceil \frac{n\, b}{a} \rceil \Leftrightarrow m_{n-1} \leq \lfloor \frac{n\, b}{a} \rfloor$, therefore $m_n \geq \lceil \frac{n\, b}{a} \rceil$, otherwise, it will remain constant.

Then we have $m_n = \min(n, \lceil \frac{n\, b}{a} \rceil)$
In the case of the reduced Syracuse sequence, we have :

For $N \geq 1$, $m_N = \lceil \frac{\text{Ln2}}{\text{Ln3}} N \rceil$

Javascript code not available in this document.

Test of m_N and JGL(N) : 65 Try

17. **Conclusion**

We have found the result of Shalom Eliahou in the case of the conjecture of the standard Syracuse sequence.

- For $v_0 > 0$ (or $b_1 > 0$), by testing the sequence up to $2^{N_0+B_1}$ (for $b_1 < 1024$ and $N_0 = 17$), we proved that the minimum length of a cycle for v is 1539
- For $v_0 < 0$ (or $b_1 < 0$), by testing the sequence up to $2^{N_0+B_1}$ (for $b_1 < 1024$ and $N_0 = 17$), we proved that the minimum length of a cycle for v is 1054

We could stop at this stage, but, in the next section, we will mathematically study some properties of the JGL transition list to retrieve this result instantly and even have a much faster way to obtain the JGL transition list.

V. Demonstration of the observations of the previous section

This section is a bit more technical and can be perfectly ignored on a first reading by focusing only on the results obtained computationally.

1. Return to the construction of JGL and the minimum length of a non-trivial cycle

How to determine the minimum length of a potential non-trivial cycle from the transition lists and v' ?

We are considering here the case $v_0 > 0$ (or $b_1 > 0$ of the first definition)

We return to the paragraph IV-4

Conversely, if we want to test whether a N, non-trivial cycle of length exists, by calculating v_0, the minimum solution that follows L_N with the algorithm of "Theorem 0", then all terms up to v_N and if $v_N = v_0$, we found a cycle. It is safe to say that it would be better to directly test all the values of $1 < v_0 < 2^N$

We can simplify things further, or at least reduce the "testing" time for each list L_N, by imposing that v_0 corresponds to the minimum value of v_n for $0 \leq n < N$.

In the case of a cycle, we therefore keep only one representative of all the circular permutations of a list, without loss of generality.

If for a list L_N, we are always in "case 0", or $\frac{3^{m_n}}{2^n} > 1$ then L_N does not correspond to a cycle since $v_N > v_0$.

In other cases, for each step where we are in the "case 1", it is necessary to calculate $V(n-1)$. If one of the values obtained is less than *maxSyr_b1* then L_N does not correspond to another cycle, otherwise, it is necessary to calculate v_N to conclude.

We can start by testing for p, the smallest value of $1 < n < N$ for which we are in the "case 1", or $\frac{3^{m_p}}{2^p} < 1 < \frac{3^{m_p}}{2^{p-1}}$ and the t_{p-1} is of "type 0".

Indeed, if $V(p-1) <$ *maxSyr_b1* , we can already conclude for L_N since $v_p < v_0$ and if this is the case for all the transition lists L_N, then, we can conclude that there is no cycle of length N, which is the result that interests us to reduce the length of a potential other cycle.
For $b_1 > 1$, we can possibly have found several trivial cycle by testing the sequence up to *maxSyr_b1*
We notice that p takes only well-defined values because m_n is incremented only if t_{n-1} is of "type 1".
Therefore, for a given m_p, we have :

$$0 < \frac{3^{m_p}}{2^p} < 1 < \frac{3^{m_p}}{2^{p-1}}$$

$$\Leftrightarrow m_p \text{Ln}3 - p\,\text{Ln}2 < 0 < m_p \text{Ln}3 - (p-1)\text{Ln}2 \text{ because the Ln function is increasing}$$

$$\Leftrightarrow \lfloor m_p \times \frac{\text{Ln}3}{\text{Ln}2} \rfloor < m_p \times \frac{\text{Ln}3}{\text{Ln}2} < p < m_p \times \frac{\text{Ln}3}{\text{Ln}2} + 1 \text{ because } m_p > 0$$

$\Leftrightarrow p = \lceil m_p \times \frac{\text{Ln}3}{\text{Ln}2} \rceil$

We must therefore calculate $\text{V}(p-1) = \frac{r_p}{1 - \frac{3^{m_p}}{2^p}}$

The denominator $1 - \frac{3^{m_p}}{2^p}$ is the same for all transition lists that have the same value p and therefore m_p according to the previous relation. It is therefore independent of the order of the $p-2$ first transitions, t_{p-1} being necessarily of "type 0".

The number of these lists is finite and infact, this number is equal to the number of ways to choose m_p transitions among $p-1$ because t_{p-1} is of "type 0".

There is therefore a list that maximizes r_p and if for this value, $\text{V}(p-1) < maxSyr\_b1$, then we can affirm that none of these lists can correspond to a cycle since $v_p < v_0$ and we can eliminate them and then consider the lists which have the first step of "case" 1", the following value of p (and if the previous reasoning is false especially in the case $b_1 \neq 1$, we can take $p+1$), ie for a value of m equal to $m_p + 1$.

If for all m_p such that $2 < p = \lceil m_p \times \frac{\text{Ln}3}{\text{Ln}2} \rceil \leq \text{N}$, (or even $p \leq \text{N}$, p being greater than the minimum length for which there is no trivial cycle) we have this property, that for the maximum value of r_p, $\text{V}(p-1) < maxSyr\_b1$, then, we can affirm that there is no cycle of less than or equal to N

To be sure that we go through all the steps p defined previously, it is enough to prioritize the transitions of "type 0", ie to make all the transitions of "type 0" possible with the constraint $v_n \geq v_0$ before adding a "type 1" transition.

Moreover, in this case, r_n is maximum for a fixed value of m_p, compared to all other potential lists, because "type 0" transitions are placed as early as possible in the transition list, respecting the constraints of the list. The demonstration of this property was made in the previous paragraph IV-10.

Thereby, we define the list of transitions which is a boundary, that we note **JGL(N)** (for "Just Greater List") as follows :

- JGL(1) = "1" or $t_0 = 1$ so that v_1 is greater than v_0 and $r_1 = \frac{b_1}{2}$
- JGL(2) = "11" or $t_1 = 1$ and therefore $r_2 = \frac{5\,b_1}{4}$ because otherwise we find the trivial cycle with $p = \lceil \frac{\text{Ln}3}{\text{Ln}2} \rceil = 2$ and then $v_0 = b_1$
- For the following steps, this algorithm is used :
  - m = 2, the number of "type 1" transitions of JGL(2)
  - For $2 \leq n < \text{N}$ (we shift the indices to have a preventive action)
    - if $\frac{3^m}{2^{n+1}} > 1 \Leftrightarrow m\,\text{Ln}3 - (n+1)\,\text{Ln}2 > 0$, then we define t_n of "type 0" and $r_{n+1} = \frac{r_n}{2}$
    - if $\frac{3^m}{2^{n+1}} < 1$, we notice that $n+1$ is one of the previous values of p because $n+1 = \lceil m \times \frac{\text{Ln}3}{\text{Ln}2} \rceil$.

      We test if we can still have a "type 0" transition. In this case JGL($n+1$), would be the list that maximizes r_p, among all the lists which have as first occurrence of $\frac{3^{m_p}}{2^p} < 1$ for $p = n+1$.

      We calculate $\text{VMax}(n) = \frac{r_n}{2} \times \frac{1}{1 - \frac{3^m}{2^{n+1}}}$ (We will use this notation in the rest of the document for this quantity) as if t_n were of "type 0". $\mathrm{VMax}(n) = \dfrac{r_{n+1}}{1 - \dfrac{3^m}{2^{n+1}}}$ with $t_n = 0$ because then

$r_{n+1} = \dfrac{r_n}{2}$

- If $\mathrm{VMax}(n) < maxSyr\_b1$ then t_n of "type 0" is not possible to respect the constraint $v_{n+1} \geq v_0$ then we define $t_n = 1$, $r_{n+1} = \dfrac{3\,r_n + 1}{2}$ and we increment m. This eliminates all lists for this value of p to have a potential cycle. By defining $t_n = 1$, JGL($n+1$) is not part of this set of lists and is the beginning of the ideal candidate for the following value of $p = \lceil m \times \dfrac{\mathrm{Ln}3}{\mathrm{Ln}2} \rceil$ because m has been incremented.
- If $\mathrm{VMax}(n) > maxSyr\_b1$ then there may be a cycle of length $n+1$, this is the minimum length for a potential cycle for the reduced Syracuse sequence v and the minimum cycle length for the standard Syracuse u is $n+1+m$. This does not mean that there is one. We leave the loop indicating the value of $n+1$ for v.

With the previous algorithm, we can deduce the minimum length of a cycle.

Initially, we had to test the 2^{N} lists of transitions... and finally, there are no more only one to test (magical !).

For N less than this minimum value of the length of a cycle, the construction of JGL(N) can already be simpler by having the property $t_n = 1$ if and only if $\dfrac{3^m}{2^{n+1}} < 1 \Leftrightarrow m\,\mathrm{Ln}3 - (n+1)\,\mathrm{Ln}2 < 0$, but it remains a step by step construction. Using the properties of JGL, we will see that it can be obtained much more quickly in blocks, by studying these patterns.

This JGL(N) boundary list also plays an important role in this approach of Syracuse theorem for $b_1 > 0$, even beyond the determination of the minimum length of a cyle, that is why we can study it further.

In case $v_0 < 0$ (or $b_1 < 0$) :

We can make an equivalent reasoning by symmetry since the list of transitions JGL is the same, it is the one that maximizes r_n for all lists of length n, this time with $v_n < v_0 < 0$.

We return to the paragraph IV-5

We then obtain the following algorithm :

- JGL(1) = "1" or $t_0 = 1$ so that v_1 is less than v_0 and $r_1 = \dfrac{b_1}{2}$. The trivial cycle with $v_0 = \text{-}b_1$ is not interesting here, since $v_0 < \text{-}maxSyr\_b1$.
- JGL(2) = "11" or $t_1 = 1$ and then $r_2 = \dfrac{5\,b_1}{4}$
- For the following steps, this algorithm is used :
  - m = 2, the number of "type 1" transitions of JGL(2)
  - for $2 \leq n <$ N (we shift the indices to have a preventive action)
    - if $\dfrac{3^m}{2^{n+1}} < 1 \Leftrightarrow m\,\mathrm{Ln}3 - (n+1)\,\mathrm{Ln}2 < 0$, it is not possible then we define t_n of "type 1", $r_{n+1} = \dfrac{3\,r_n + b_1}{2}$, JGL($n+1$) = JGL($n$) + "1" and we increment m
    - if $\dfrac{3^m}{2^{n+1}} > 1$, there is always v_0 checking the inequality then $t_n = 0$, $r_{n+1} = \dfrac{r_n}{2}$ and JGL($n+1$) = JGL(n) + "0"

      Only, as $r_{n+1} = \dfrac{r_n}{2}$ and $v'_{n+1} < v_0$, to have a cycle of length $n+1$, it is imperative that

$r_{n+1} > 0$ must be large enough so that $v_{n+1} = v'_{n+1} + r_{n+1} \geq v_0$, equality would mean a cycle.

Therefore, a necessary condition to have a cycle, is $\mathrm{VMax}(n) \leq v_0 < 0$, the previous inequality in the other direction.

- If $\mathrm{VMax}(n) > \text{-}maxSyr\_b1$ then it would be a trivial cycle
- If $\mathrm{VMax}(n) < \text{-}maxSyr\_b1$ then there may be a cycle of length $n+1$, this is the minimum length for a potential cycle for the reduced Syracuse sequence v and the minimum length of a cycle for the standard Syracuse sequence u is $n+1+m$. This does not mean that there is one. We leave the loop indicating the value of $n+1$ for v.

With the previous algorithm, we can deduce the minimum length of a cycle.

2. **Patterns of JGL(n)**

We prove here that the JGL(n) list has repetitive patterns and this will allow us to construct it in blocks, faster than in the previous calculation.

We consider the case where $n < 10439860590$ therefore the transition t_n is of "type 0" if is only if $\frac{3^{m_n}}{2^n} \geq 2$.

For $b_1 > 1$ and $v_0 > 0$, the bound is $n < 1538$
For $b_1 > 1$ and $v_0 < 0$, the bound is $n < 1053$

Next (for $n \geq 10439860590$), indeed, there could be a permutation of two successive transitions quite rarely, which would obviously distort JGL(n) but would have absolutely no impact on the order of magnitude of r_n except for the only values of n affected (see remarks IV-10). And then, if we prove by the calculation or the reasoning that the sequence is verified for greater values of v_0 then the bound will be even greater... and if, finally, there are no other cycles, then the bound no more exists (infinity).

The previous reasoning made in the case of standard Syracuse sequence is also valid for the generalization. As we are going to prove that there are no other cycles than the trivial cycles, we can act as if the limit does not exist even though it is a bit premature.

If $\frac{m}{d}$ is close to X by excess, then $\frac{3^m}{2^{\mathrm{N}}}$ (with $\mathrm{N} = m + d$) is close to 1 by excess because $\frac{3^m}{2^{\mathrm{N}}} = \mathrm{e}^{d(\mathrm{Ln3} - \mathrm{Ln2})(m/d - X)}$ then we can clearly sense that the N transitions (called "pattern") can be repeated a number of times as long as the "type 1" transition (for $0 < n < \mathrm{N}$), the most likely to turn into "type 0" has not mutated yet. This breakdown seems more suitable to me than for the approximations of X by defect.

We can set notations for operations on lists :

- $\mathrm{L}_1 + \mathrm{L}_2$: The concatenation of L_1 and L_2
- $p\,\mathrm{L}_1$: p times the concatenation of L_1, or $p\,\mathrm{L}_1 = \mathrm{L}_1 + ... + \mathrm{L}_1$, p times

By the way, we can see it for JGL(kN) for N = 1054 and $k = 200$:
$\mathrm{L}_1 = \mathrm{JGL}(1054)$
$\mathrm{JGL}(200\times1054) = \mathrm{JGL}(210800) = 24\times\mathrm{L}_1 + \mathrm{L}_2 + 23\times\mathrm{L}_1 + \mathrm{L}_2 + 24\times\mathrm{L}_1 + \mathrm{L}_2 + 23\times\mathrm{L}_1 + \mathrm{L}_2 + 24\times\mathrm{L}_1 + \mathrm{L}_2 + 23\times\mathrm{L}_1 + \mathrm{L}_2 + 23\times\mathrm{L}_1 + \mathrm{L}_2 + 24\times\mathrm{L}_1 + \mathrm{L}_2 + 8\times\mathrm{L}_1 + \mathrm{L}_3$

Description of the number of pattern $\mathrm{L}_1 = \mathrm{JGL}(1054)$:

number of times	number of occurences	index of occurences
24	4	0,50508,101016,176251
23	4	25781,76289,126797,151524
8	1	202032

Description of the elements L_i:

L_i	length	number of occurences	value index of occurences
L_2	485	8	JGL(484) + "0" 25296,50023,75804,100531,126312,151039,175766,201547
L_3	336	1	JGL(336) 210464

And you can test it (better for N = 19, 84, 569 or 1054 than for 149 for example) :

Value of k : 200 , Value of N : 1054 Try See/Hide code

But let's prove it :

Determination of the number of "patterns" for an approximation by excess of X :

Let $\frac{m}{d}$ (with $\mathrm{N} = m + d$), an excess approximation of X (see §IV-9), then $F_{\mathrm{N}} = \frac{3^m}{2^{\mathrm{N}}}$ is the minimum for $0 < n \leq \mathrm{N}$ of $F_n = \frac{3^{m_n}}{2^n} > 1$, since for all the previous excess approximations of X, this value is greater and we do not take into account values lower than 1 ($F_n < 1$).

Here, we will call "pattern", the list of transitions JGL(N) (or character string)

Reminder : We are in the case where the transition t_n is of "type 0" if is only if $F_n = \frac{3^{m_n}}{2^n} \geq 2$ (or $\frac{F_n}{2} = \frac{3^{m_n}}{2^{n+1}} > 1$) because $n < 10439860590$ (or 1538, or 1053) represents the position $n+1$ of the JGL list.

The transition that has the best chance of "shifting" (or changing type, switching from a transition from "type 1" to transition of "type 0") is the "type 1" transition which is closest to the condition of having a "type 0" transition.

Therefore we sort in descending order the values $F_n = \frac{3^{m_n}}{2^n} < 2$ for $0 \leq n < \mathrm{N}$.

It is immediate that the first values correspond to the values of $n+1$ such as $\frac{m_n}{n - m_n + 1}$ always is an approximation of X for the same reasons.

Let a, the index corresponding to the maximum and b, the index corresponding to the second value with $b < a$
We are looking for q_1, the smallest integer value, such as : $F_{q_1 \mathrm{N} + a} = (F_{\mathrm{N}})^{q_1} F_a \geq 2$
As long as a transition does not "shift" (or "change") i.e. the pattern JGL(N) repeats, we always have $F_{p\mathrm{N}+n} = (F_{\mathrm{N}})^p F_n$, for all $p \geq 0$ and all $0 \leq n < N$

$$\Leftrightarrow q_1 \mathrm{Ln}(F_{\mathrm{N}}) + \mathrm{Ln}(F_a) \geq \mathrm{Ln}(2)$$
$$\Leftrightarrow q_1 \geq \frac{\mathrm{Ln}(2) - \mathrm{Ln}(F_a)}{\mathrm{Ln}(F_{\mathrm{N}})}$$

Therefore, $q_1 = \left\lceil \frac{\mathrm{Ln}(2) - \mathrm{Ln}(F_a)}{\mathrm{Ln}(F_{\mathrm{N}})} \right\rceil$

We will therefore have JGL($q_1 \mathrm{N} + a$) = q_1 JGL(N) + JGL($a - 1$) + "0" and $F_{q_1 \mathrm{N} + a} = \frac{(F_{\mathrm{N}})^{q_1} F_a}{2}$
For example, for N = 1054, we have $a = 485$ and $q_1 = 24$, which gives JGL($24 \times 1054 + 485$) = JGL(25781) = 24×JGL(1054) + JGL(484) + "0". It should be noted that 25781 is the following approximation of X by excess

after 1054, by a misuse of language if we consider the sum of the numerator and the denominator.

We can test for N = 1054 Try See/Hide code

Then, the pattern JGL(N) will repeat q_2 times until we find $0 \leq n < N$ such that $\frac{(F_{\mathrm{N}})^{q_1} F_a}{2}(F_{\mathrm{N}})^{q_2} F_n > 2$

The minimum of q_2 is reached for $n = a$ but it could be that the same value q_2 is obtained for $n = b$ (or even others) and as $b < a$, it would be this transition that would "shift" (or "change") first (for $n > a$, it has no impact because the transition a would still shift before.

If $\frac{F_a}{F_b} > F_{\mathrm{N}}$ then it's not possible that b change before a (and a fortiori the others) and then it would necessarily always be the transition a which shifts, for q_2 but also for q_k for all $k \geq 1$.

We can simply note it for the values of N that interest us (corresponding to the approximations of X by excess between $19 \leq \mathrm{N} \leq 1082233$) with the previous test code.

For example, in the previous test, we see that for N = 1054, $\frac{F_a}{F_b} = \frac{F_{485}}{F_{401}} \approx 1.0020465696\, F_{1054} > F_{1054} = F_{\mathrm{N}}$.

In fact, we can prove that $\frac{F_a}{F_b} > F_{\mathrm{N}}$, that is what we do right away, before determining q_2, to be perfectly rigorous.

Let's prove that : $\frac{F_a}{F_b} > F_{\mathrm{N}}$

$\Leftrightarrow \mathrm{Ln}(F_a) - \mathrm{Ln}(F_b) > \mathrm{Ln}(F_{\mathrm{N}})$

$\Leftrightarrow d_a(\mathrm{Ln}3 - \mathrm{Ln}2)\left(\frac{m_a}{d_a} - x\right) - d_b(\mathrm{Ln}3 - \mathrm{Ln}2)\left(\frac{m_b}{d_b} - x\right) > d_{\mathrm{N}}(\mathrm{Ln}3 - \mathrm{Ln}2)\left(\frac{m_{\mathrm{N}}}{d_{\mathrm{N}}} - x\right)$

$\Leftrightarrow d_a\left(\frac{m_a}{d_a} - x\right) - d_b\left(\frac{m_b}{d_b} - x\right) > d_{\mathrm{N}}\left(\frac{m_{\mathrm{N}}}{d_{\mathrm{N}}} - x\right)$

$\Leftrightarrow X > \frac{m_a - m_b - m_{\mathrm{N}}}{d_a - d_b - d_{\mathrm{N}}}$ car $d_a - d_b - d_{\mathrm{N}} < 0$

$\Leftrightarrow X > \frac{m_{\mathrm{N}} + m_b - m_a}{d_{\mathrm{N}} + d_b - d_a}$

By construction with the Stern-Brocot tree, at each step, we keep the best approximation of X (by defect or by excess) so the median is always a new bound of the interval in which X is.

As $b < a < \mathrm{N}$, then, by constructing the median, for the steps between that of b and that of a, there are only approximations by excess, then that of a and then only approximations by excess until that of N which is just a particular value.

At the step corresponding to b, defect approximation, we have $\frac{m_b}{d_b} < X < \frac{p}{q}$, $\frac{p}{q}$ doesn't matter.

In the k next steps before that of a, we have approximations by excess therefore $\frac{m_b}{d_b} < X < \frac{k\, m_b + p}{k\, d_b + q}$

The step $k+1$ corresponds to a and $= \frac{(k+1) m_b + p}{(k+1) d_b + q}$ and $\frac{m_a}{d_a} < X < \frac{k\, m_b + p}{k\, d_b + q} = \frac{m_a - m_b}{d_a - d_b}$

In the l following steps until that of N, we have excess approximations therefore $\frac{m_a}{d_a} < X < \frac{(l+1) m_a - m_b}{(l+1) d_a - d_b}$

And so there exists $l > 0$ such as $\frac{m_{\mathrm{N}}}{d_{\mathrm{N}}} = \frac{(l+1)m_a - m_b}{(l+1)d_a - d_b}$

Now let's calculate, $\frac{m_{\mathrm{N}}+m_b-m_a}{d_{\mathrm{N}}+d_b-d_a} = \frac{(l+1)m_a - m_b + m_b - m_a}{(l+1)d_a - d_b + d_b - d_a} = \frac{l\,m_a}{l\,d_a} = \frac{m_a}{d_a} < X$ because it is a defect approximation of X

Conclusion : $\frac{F_a}{F_b} > F_{\mathrm{N}}$

Let's resume the search for q_2 knowing that it has been proven that it was always the transition a which shifts.

Thus, to find q_2, we are looking for the smallest integer value such as $(F_{\mathrm{N}})^{q_1} \times \frac{F_a}{2} (F_{\mathrm{N}})^{q_2} F_a \geq 2$

$\Leftrightarrow (F_{\mathrm{N}})^{q_1}(F_{\mathrm{N}})^{q_2}(F_a)^2 \geq 2^2$

$\Leftrightarrow q_1 \mathrm{Ln}(F_{\mathrm{N}}) + q_2 \mathrm{Ln}(F_{\mathrm{N}}) + 2\,\mathrm{Ln}(F_a) \geq 2\,\mathrm{Ln}2$

$\Leftrightarrow q_2 = \left\lceil \frac{2(\mathrm{Ln}2 - \mathrm{Ln}(F_a)) - q_1 \mathrm{Ln}(F_{\mathrm{N}})}{\mathrm{Ln}(F_{\mathrm{N}})} \right\rceil$

We will therefore have JGL($(q_1+q_2)\mathrm{N}+2a$) = q_1 JGL(N) + JGL($a-1$) + "0" + q_2 JGL(N) + JGL($a-1$) + "0"

and $F_{(q_1+q_2)\mathrm{N}+2a} = \frac{(F_{\mathrm{N}})^{q_1+q_2}(F_a)^2}{2^2}$

For example, for N = 1054, we have a = 485, q_1 = 24 and q_2 = 23, which gives JGL($(24+23)\times1054+2\times485$) = JGL(50508) = 24×JGL(1054) + JGL(484) + "0" + 23×JGL(1054) + JGL(484) + "0". It should also be noted that 50508 is the approximation of X by excess following 25781.

Let us show by induction that :

$q_k = \left\lceil \frac{k(\mathrm{Ln}2 - \mathrm{Ln}(F_a)) - \sum_{i=1}^{k-1} q_i \mathrm{Ln}(F_{\mathrm{N}})}{\mathrm{Ln}(F_{\mathrm{N}})} \right\rceil$ and therefore JGL$\left(\left(\sum_{i=1}^{k} q_i \mathrm{N}\right) + k\,a\right) = \sum_{i=1}^{k}$ (q_i×JGL(N) + JGL($a-1$) + "0")

The formula is verified for k = 1 and k = 2

We assume it's true for k and we prove that it is true for $k+1$.

To find q_{k+1}, we are looking for the smallest integer value such as $\sum_{i=1}^{k} (F_{\mathrm{N}})^{q_i} \times \frac{(F_a)^k}{2^k} (F_{\mathrm{N}})^{q_{k+1}} F_a \geq 2$ because $\frac{F_a}{F_b} > F_{\mathrm{N}}$

$\Leftrightarrow \sum_{i=1}^{k} q_i \mathrm{Ln}(F_{\mathrm{N}}) + k\,\mathrm{Ln}(F_a) - k\,\mathrm{Ln}2 + q_{k+1} \mathrm{Ln}(F_{\mathrm{N}}) + \mathrm{Ln}(F_a) \geq \mathrm{Ln}2$

$\Leftrightarrow q_{k+1} = \left\lceil \frac{(k+1)(\mathrm{Ln}2 - \mathrm{Ln}(F_a)) - \sum_{i=1}^{k} q_i \mathrm{Ln}(F_{\mathrm{N}})}{\mathrm{Ln}(F_{\mathrm{N}})} \right\rceil$ i.e. the property at rank $k+1$, which ends the proof.

Conclusion: we can get JGL(n) in blocks, in a much faster way since we can find by calculation the patterns that had been observed at the beginning of this paragraph.

We can calculate JGL(n) in block with N = 1054 et n = 301994 Try See/Hide code

3. **Quick calculation of $R_0 = r_{\mathrm{N}_0}$ for JGL(N_0)**
    1. **Calculation of r_{N} for a concatenation of transition lists**

       In this paragraph, we use the following notations :

- N_0 has nothing to do in this paragraph with the value corresponding to the verification of the sequence
- $L_k(N_k, m_k, d_k)$, a list of transitions of length N_k with m_k transitions of "type 1" et d_k transitions of "type 0", therefore $N_k = m_k + d_k$
- $F_k = \frac{3^{m_k}}{2^{N_k}}$, the multiplicative factor of $L_k(N_k, m_k, d_k)$
- $R_k = r_{N_k}$, the "residue" at the end of the transition list $L_k(N_k, m_k, d_k)$
- $f_k = \prod_{i=1}^{k} \frac{1}{F_i}$

Therefore, $v_{(N_k, L_k)} = F_k v_{(0, L_k)} + R_k$ characterizes $L_k(N_k, m_k, d_k)$ which defines the N_k low-order bits of $v_{(0, L_k)}$, see "Theorem 0" (paragraph VI-2

We can set notations for operations on lists :
- $L_1 + L_2$: The concatenation of L_1 and L_2
- $p L_1$: p times the concatenation of L_1, or $p L_1 = L_1 + ... + L_1$, p times

a. **Case $L_0 = L_1 + L_2$:**

We have:
- $N_0 = N_1 + N_2$, $m_0 = m_1 + m_2$, $d_0 = d_1 + d_2$
- $v_{(N_1, L_1)} = F_1 v_{(0, L_1)} + R_1$
- $v_{(N_2, L_2)} = F_2 v_{(0, L_2)} + R_2$

By replacing $v_{(0, L_2)}$ by $v_{(N_1, L_1)}$, we get :

$v_{(N_0, L_0)} = F_2(F_1 v_{(0, L_0)} + R_1) + R_2$, the N_1 low-order bits of $v_{(0, L_0)}$ being defined by L_1 and the N_2 next bits by L_2 knowing that these are not necessarily (and even generally) the same as for $v_{(0, L_1)}$ but the calculation of $v_{(0, L_0)}$ and $v_{(N_0, L_0)}$ matters little to us, it is the value of R_0 that interests us here.

$v_{(N_0, L_0)} = F_0 v_{(0, L_0)} + R_0$ with $R_0 = F_2 R_1 + R_2 = \frac{3^{m_0}}{2^{N_0}}\left(\frac{R_1}{F_1} + \frac{R_2}{F_1 F_2}\right) = \frac{3^{m_0}}{2^{N_0}}(f_1 R_1 + f_2 R_2)$ with $f_k = \prod_{i=1}^{k} \frac{1}{F_i}$

b. **Case $L_0 = L_1 + L_2 + L_3$:**

In the same way, since $v_{(N_3, L_3)} = F_3 v_{(0, L_3)} + R_3$, by replacing $v_{(0, L_3)}$ with the value of $v_{(N_0, L_0)}$ previously obtained for $L_0 = L_1 + L_2$, we get :

$v_{(N_0, L_0)} = F_0 v_{(0, L_0)} + R_0$ with $R_0 = F_2 F_3 R_1 + F_3 R_2 + R_3 =$

$$\frac{3^{m_0}}{2^{N_0}}\left(\frac{R_1}{F_1} + \frac{R_2}{F_1 F_2} + \frac{R_3}{F_1 F_2 F_3}\right) = \frac{3^{m_0}}{2^{N_0}}(f_1 R_1 + f_2 R_2 + f_3 R_3) = \frac{3^{m_0}}{2^{N_0}} \sum_{k=1}^{3} f_k R_k$$

c. **General case $L_0 = \sum_{k=1}^{n} L_k$:**

By immediate induction, we have $R_0 = \frac{3^{m_0}}{2^{N_0}} \sum_{k=1}^{n} f_k R_k$

d. **Special case $L_0 = n\,L_1$:**

We have

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \sum_{k=1}^{n} f_k R_1 = \frac{3^{m_0}}{2^{N_0}} R_1 \sum_{k=1}^{n} \left(\frac{1}{F_1}\right)^k = R_1 \sum_{k=1}^{n-1} \left(\frac{3^{m_0}}{2^{N_0}}\left(\frac{1}{F_1}\right)^k\right) = R_1 \sum_{k=1}^{n-1} \left((F_1)^n \times \frac{1}{(F_1{}^k)}\right)$$

because $m_0 = n\,m_1$ and $N_0 = n\,N_1$

$R_0 = R_1 \sum_{k=1}^{n-1} (F_1)^{n-k} = R_1 \sum_{k=0}^{n-1} (F_1)^k$ by changing the indexes.

Then, we have a sum of terms of a geometric sequence of first term 1 and factor F_1

$$R_0 = \frac{1-(F_1)^n}{1-F_1} R_1$$

Moreover, if $F_1 \approx 1$, by setting $u_1 = 1 - F_1$ then $(F_1)^n \approx 1 - n\,u_1$ and $R_0 \approx n\,R_1$, which is a linear approximation.

e. **General case $L_0 = \sum_{k=1}^{n} p_k L_k + L_{n+1}$:**

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \left(\sum_{i=0}^{p_1-1} \frac{R_1}{(F_1)^{i+1}} + \frac{1}{(F_1)^{N_1}} \sum_{i=0}^{p_2-1} \frac{R_2}{(F_2)^{i+1}} + \ldots\right) + R_{n+1}$$

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \left(\frac{R_1}{(F_1)^{p_1}} \sum_{i=0}^{p_1-1} (F_1)^i + \frac{R_2}{(F_1)^{p_1}(F_2)^{p_2}} \sum_{i=0}^{p_2-1} (F_2)^i + \ldots\right) + R_{n+1}$$

or more mathematically :

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \left(\sum_{k=1}^{n} \frac{R_k}{\prod_{j=1}^{k} (F_j)^{p_j}} \sum_{i=0}^{p_k-1} (F_k)^i\right) + R_{n+1}$$

By replacing the sums of terms in geometric progression, we get :

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \left(\sum_{k=1}^{n} \frac{R_k}{\prod_{j=1}^{k} (F_j)^{p_j}} \times \frac{1-(F_k)^{p_k}}{1-F_k}\right) + R_{n+1}$$

Moreover, if $F_k \approx 1$, for all $1 \le k \le n$, by setting $u_k = 1 - F_k$ then $(F_k)^{p_k} \approx 1 - p_k u_k$, and in this case :

$$R_0 \approx \frac{3^{m_0}}{2^{N_0}} \left(\sum_{k=1}^{n} \frac{p_k R_k}{\prod_{j=1}^{k} (F_j)^{p_j}}\right) + R_{n+1}$$

In some special cases $\prod_{j=1}^{k} (F_j)^{p_j} \approx 1$ for all $1 \le k \le n$ and in this case :

$$R_0 \approx \frac{3^{m_0}}{2^{N_0}} \left(\sum_{k=1}^{n} p_k R_k\right) + R_{n+1}$$

f. **Calculation test of R_0 :**

Number of lists : 5

Maximum length for each list : 19 (for 19 or 84, F_k is always close to 1)

☐ Identical lists

Try | See/Hide code

2. **Application for JGL**

According to the paragraph V-3-1-e, we have a general formula for calculating R_0 in the case where $L_0 = \sum_{k=1}^{n} p_k L_k + L_{n+1}$

NB : The value N_0 has nothing to do in this paragraph with the value corresponding to the verification of the sequence.

The formula is as follows :

$$R_0 = \frac{3^{m_0}}{2^{N_0}} \left(\sum_{k=1}^{n} \frac{R_k}{\prod_{j=1}^{k} (F_j)^{p_j}} \times \frac{1-(F_k)^{p_k}}{1-F_k} \right) + R_{n+1}$$

According to the previous paragraph, JGL(N_0) is composed of alternating identical lists of transitions, the last list being a remainder, therefore :

- For $1 < 2i+1 \le n$, $L_{2i+1} = L_1$, odd index lists are all L_1
- For $2 < 2i \le n$, $L_{2i} = L_2$, even index lists are all L_2
- $N_2 < N_1$
- $N_{n+1} < N_1$, L_{n+1} is the beginning of L_1
- $F_1 > 1$ and $F_1 \approx 1$ because N_1 corresponds to an "approximation" of X by excess
- $F_2 < 1$ and $F_2 \approx 1$ because N_2 corresponds to an "approximation" of X by defect
- $(F_1)^{p_{2i+1}} (F_2)^{p_{2i+2}} \approx 1$, often a compensation with $\left|p_{2i+1} - p_1\right| \le 1$ et $p_{2i+2} = 1$
- $R_2 \approx \frac{N_2}{N_1} R_1$

Under these conditions, we have the approximation that $R_0 \approx \frac{3^{m_0}}{2^{N_0}} \times \frac{N_0 - N_{n+1}}{N_1} R_1 + R_{n+1} = \frac{3^{m_0}}{2^{N_0}} \times \frac{N_0}{N_1} R_1 + R_{n+1} - \frac{3^{m_0}}{2^{N_0}} \times \frac{N_{n+1}}{N_1} R_1$, for N_0 large enough, which interests us.

Moreover, the maximum value of $r_n - \frac{3^{m_n}}{2^n} \times \frac{n}{N_1} R_1 \approx b_1$ for $0 \le n < N_1$.

For example, this difference is equal to $1.18\, b_1$ for $N_1 = 1054$, so

$R_0 \approx \frac{3^{m_0}}{2^{N_0}} \times \frac{N_0}{N_1} R_1$ which is a linear approximation.

This amounts to saying that permuting a few transitions in JGL(N_0) does not have a significant impact on the value of R_0

By considering JGL(n), for all n such as t_n is of "type 1", we have $1 < F_n < 2$ therefore $\frac{m_n}{d_n + 1} < X < \frac{m_n}{d_n}$

or $m_n - X < d_n X < m_n$ then $m_n \approx d_n X$ for n large enough,

And therefore $n = m + d \approx (1+X)d$

And then, after simplification by $1+X$, we have the following property :

1. **Property V-3-2-1 : For JGL(N_0), $R_0 = r_{N_0} \approx \frac{3^{m_0}}{2^{N_0}} \times \frac{N_0}{N_1} R_1 \approx \frac{3^{m_0}}{2^{N_0}} \times \frac{d_0}{d_1} R_1$**

3. If N_0 is such as t_{N_0} is of "type 1", always with $N_0 < 10439860590$ or 1538, or 1053

   In this case $F_{N_0} = \frac{3^{m_0}}{2^{N_0}} < 2$ and $R_1 < 0.45\, d_1 b_1$ (0.4113 for $N_1 = 1054$, for example)

   and then $r_n < (d_n + 1) b_1$ seems a correct upper bound.

   This is verified in the test code of IV-13 for $n < 10000$, and then $2 \times 0.45 = 0.9 < 1$

   Property V-3-3 : For all $n > 0$, in JGL(n), if t_n is of "type 1", then $r_n < (d_n + 1) b_1$

   The values of R_0 are given for $b_1 = 1$, they should be multiplied by b_1 in the general case

   We can calculate R_0 in block with avec N = 1054 et n = 140665 Try
   See/Hide code

4. If $N_0 + 1$ corresponds to an "approximation" of X, ie $\frac{m_0}{d_0 + 1}$ is an approximation of X

   then it is a special case of the previous case, $\frac{3^{m_0}}{2^{N_0}} \approx 2$ and $R_{n+1} \approx 2 R_1$ and therefore

   $R_0 \approx 2 \times \frac{N_0}{N_1} R_1 \approx 2 \times \frac{d_0}{d_1} R_1 \approx 2 \times 0.411\, d_0 b_1 = 0.822\, d_0 b_1$, linear approximation, which allows to have the values of the paragraph IV-13 and of the paragraph IV-14instantly, whereas it required hours, or even days of calculations, step by step. We find results similar to the errors of precision.

   Property V-3-4 : For all $n+1$ corresponding to an "approximation" of X, in JGL(n), $r_n \approx$ 0.822103 $d_n b_1$

   The values of R_0 are given for $b_1 = 1$, they should be multiplied by b_1 in the general case

   We can calculate R_0 in block for the defect approximations : with N = 1054 and $N_0 + 1$ = 10439860591 Try

   We can calculate R_0 in block for the excess approximations : with N = 1054 and $N_0 + 1$ = 93328606422 Try

4. **Calculation of VMax(n) for $n+1$ corresponding to an "approximation" of X**

   As a reminder, $\text{VMax}(n) = \frac{2^n r_n}{2^n - 3^{m_n}}$ is the maximum value of v_0 satisfying **(In-1)**, which is $\left(1 - \frac{3^{m_n}}{2^{n+1}}\right) v_0 \leq \frac{r_n}{2}$

   with $\frac{3^{m_n}}{2^{n+1}} < 1$ for $v_0 > 0$ and with $\frac{3^{m_n}}{2^{n+1}} > 1$ for $v_0 < 0$

   We consider the case where $\frac{m_n}{d_n + 1}$ is an approximation of X .

   In this case, $\text{VMax}(n)$ represents the maximum value of v_0 for which there may be a cycle of length $n+1$.

   Thereby, if the sequence is verified up to this value, then the minimum length of a cycle is equal to the "next value" of $n+1$

   Then, according to the property IV-7-1, $\frac{3^{m_n}}{2^{n+1}} = \frac{3^{m_n}}{2^{N_n + d_n + 1}} = e^{(d_n + 1)(\text{Ln}3 - \text{Ln}2)(m_n/(d_n+1) - X)} = e^u$

But, $\frac{m_n}{(d_n+1)} - X$ is on the order of $\frac{1}{(d_n+1)^2}$, then $u \approx 0$.

Therefore, we can use the approximation $e^u \approx 1+u$ when $u \approx 0$.

And $1 - \frac{3^{m_n}}{2^{n+1}} \approx 1-(1+u) = -u$

From inequality **(In-1)**, we therefore have :

$-(d_n+1)(\mathrm{Ln}3-\mathrm{Ln}2)\left(\frac{m_n}{d_n+1} - X\right)\mathrm{VMax}(n) \approx \frac{r_n}{2} \approx \frac{1}{2} \times \frac{3^{m_n}}{2^n} \times \frac{d_n}{d_1} R_1$ with R_1 and d_1 for N_1 corresponding to an "approximation" by excess of X according to the previous paragraph.

As $d_n+1 \approx d_n$ and $\frac{3^{m_n}}{2^n} \approx 2$, after simplification,

- **Property V-4-1 : VMax(n) $\approx \frac{R_1}{d_1} \times \frac{1}{\mathrm{Ln}3-\mathrm{Ln}2} \times \frac{1}{X - \frac{m_n}{d_n+1}} \approx 1.0136 \times \frac{b_1}{X - \frac{m_n}{d_n+1}}$ for $n+1$, "approximation" of X**

<u>Note</u> : We can also take as the value of N_1, values that "correspond to defect approximations" of X by changing the last transition of the pattern JGL(N_1) for the calculation of R_1. For example, for $N_1 = 301994$, we get interesting results because the precision $x - \frac{190537}{111457} \approx 1.4274 \times 10^{-12}$ or $(x - \frac{190537}{111457}) \times 111457^2 \approx 1.7732 \times 10^{-2}$

We find the results of IV-13 for $v_0 > 0$ and of IV-14 for $v_0 < 0$ near to truncation errors.

Here, the calculation certainly gives more accurate results for large values of n and especially instantly.

With $N_1 = 1054$, we especially get :

- $\mathrm{VMax}(485) = 2^{16.14} b_1$
- $\mathrm{VMax}(569) = -2^{16.3} b_1$ for $v_0 < 0$
- $\mathrm{VMax}(301993) = 2^{39.37} b_1$
- $\mathrm{VMax}(10439860590) = 2^{67.09} b_1$
- $\mathrm{VMax}(114208327603) = 2^{71.41} b_1$.

We will set :

- $N = n+1$, length of a cycle for v
- $N_u = N + m_n$, length of a cycle for u
- $v_0 = \mathrm{VMax}(n)$, the maximum value for which there may be a cycle of this length

The values of R_0 are given for $b_1 = 1$, they should be multiplied by b_1 in the general case

For values of N corresponding to "approximations of X" by defect :

We can calculate $\mathrm{VMax}(n)$ directly with N_1 = 1054 Try See/Hide code

For values of N corresponding to "approximations of X" by excess :

We can calculate $\mathrm{VMax}(n)$ directly with N_1 = 1054 Try

5. **Proof of the experimental result of the paragraph IV-13 for $v_0 > 0$:**

We could be satisfied with the observation of the previous paragraph IV-13 regarding the rows of the table of results, but we seek to prove it here.

We consider the inequality, called **(In-1)**, $\left(1-\frac{3^{m_n}}{2^{n+1}}\right)v_0 \le \frac{r_n}{2}$ with $\frac{3^{m_n}}{2^{n+1}} < 1$

The goal is to prove that $\frac{m_n}{d_n+1}$ is a defect approximation of X (defined in paragraph IV-8 with Stern-Brocot tree) is a necessary and sufficient condition for the maximum value of v_0, called VMax(n), satisfying the inequality **(In-1)** increases.

Thanks to patterns of JGL(n), we already have could very quickly calculate r_n and also VMax (n).

We remind you that $\frac{3^{m_n}}{2^{n+1}} = \frac{3^{m_n}}{2^{m_n+d_n+1}} = \mathrm{e}^{(d_n+1)(\mathrm{Ln}3-\mathrm{Ln}2)(m_n/(d_n+1)-X)}$

Therefore $\frac{3^{m_n}}{2^{n+1}} < 1 \Leftrightarrow \frac{m_n}{d_n+1} < X$ because $\mathrm{e}^u < 1 \Leftrightarrow u < 0$, what links the powers and the fractions that approximate X, what often be used.

And **(In-1)** $\Leftrightarrow \left(1-\frac{3^{m_n}}{2^{n+1}}\right)v_0 \le \frac{r_n}{2}$ with $\frac{m_n}{d_n+1} < X$

If appX (k) = $\frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ with $\frac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}} < 1$ and $\mathrm{NX}_k = \mathrm{mX}_k+\mathrm{dX}_k$ is the k$^{\mathrm{th}}$ approximation of X par défaut, by defect, defined in IV-8, then let's prove :

- **The existence of the step *n* in the construction of JGL(*n*) such that $\frac{m_n}{d_n+1}$ = appX(*k*)**

  As $n = m_n+d_n$ increases from 1 in 1 (increment) in the construction of JGL(n), then the number of steps of "type 1", which represents the numerator of $\frac{m_n}{d_n}$ also takes all values.

  Then, there exists a step with $\mathrm{N}_{\mathrm{n1}} = m_{\mathrm{n1}}+d_{\mathrm{n1}}$ such that $m_{\mathrm{n1}} = \mathrm{mX}_k$ and for all $n < \mathrm{N}_{\mathrm{n1}}$, $m_n < m_{\mathrm{n1}}$.

  Then, in construction, the number of "type 1" steps does not vary as long as a "type 0" transition is possible, therefore we have all the steps $n = m_{\mathrm{n1}}+d_n$ with $d_n = d_{\mathrm{n1}}+i$ and $0 \le i \le p$ where p is the maximum value such that $\frac{3^{m_1}}{2^{m_{\mathrm{n1}}+d_{\mathrm{n1}}+i+1}} \ge 1$ therefore for $i = p$, we still have $\frac{3^{m_1}}{2^{m_{\mathrm{n1}}+d_{\mathrm{n1}}+p+1}} \ge 1$ (and therefore $\frac{m_1}{d_{\mathrm{n1}}+p+1} > X$, X not being rational), therefore the transition of "type 0" is possible and therefore the step n with $m_n = m_{\mathrm{n1}}$ and $d_n = d_{\mathrm{n1}}+p+1$ is part of the construction of JGL.

  However $\frac{3^{m_1}}{2^{d_{\mathrm{n1}}+p+2}} = \frac{3^{m_n}}{2^{n+1}} < 1$, by definition of p, and therefore $\frac{m_1}{d_{\mathrm{n1}}+p+2} < X$.

  Therefore, $\frac{m_1}{d_{\mathrm{n1}}+p+2} < X < \frac{m_1}{d_{\mathrm{n1}}+p+1}$ and then by identification, necessarily $d_{\mathrm{n1}}+p+2 = \mathrm{dX}_k$ and we indeed have the step $n = m_n+d_n$ with $m_n = m_{\mathrm{n1}} = \mathrm{mX}_k$ and $d_n = d_{\mathrm{n1}}+p+1$ which is a JGL step, such that $\frac{m_n}{d_n+1}$ = appX (k)

- **Sufficient condition** : VMax($\mathrm{mX}_k+(\mathrm{dX}_k-1)$) > VMax($n$) for all $n < \mathrm{mX}_k+\mathrm{dX}_k-1$

  - $r_{\mathrm{mX}_k+\mathrm{dX}_k-1} > r_n$ for all $n < \mathrm{mX}_k+\mathrm{dX}_k-1$ because according to the property V-3-2-1 ($R_0 = r_{\mathrm{N}_0} \approx \frac{3^{m_0}}{2^{\mathrm{N}_0}} \times \frac{d_0}{d_1} R_1$), r_n is proportional to d_n and $\frac{3^{m_n}}{2^n}$.

However, $\mathrm{dX}_k \geq d_n$ for all $0 < n < \mathrm{NX}_k$ because it is the last step

and $\dfrac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}} \geq \dfrac{3^{m_n}}{2^n}$ for all $0 < n < \mathrm{NX}_k$ because $X - \dfrac{\mathrm{mX}_k}{\mathrm{dX}_k} < X - \dfrac{m_n}{d_n}$ since it is the best defect approximation of the Stern-Brocot tree

- $X - \dfrac{\mathrm{mX}_k}{\mathrm{mX}_k + \mathrm{dX}_k} < X - \dfrac{m_n}{n - m_n + 1}$ for all $n < \mathrm{mX}_k + \mathrm{dX}_k - 1$, by definition of appX (k)

  therefore $1 - \dfrac{3^{\mathrm{mX}_k}}{2^{\mathrm{mX}_k + \mathrm{dX}_k}} < 1 - \dfrac{3^{m_n}}{2^{\mathrm{N}_n + 1}}$ for all $n < \mathrm{mX}_k + \mathrm{dX}_k - 1$

The conclusion is therefore immediate, the numerator being the largest ever reached and the smallest denominator ever reached.

- **Necessary condition**

Here, we prove that VMax(n) is solely related to the quantity $X - \dfrac{m_n}{d_n + 1} > 0$ and therefore a necessary condition for VMax(n) to increase, is that the $\dfrac{m_n}{d_n + 1}$ correspond to the defect approximations of X or appX (k) for $k \geq 0$

Either, for all $\mathrm{NX}_k < n < \mathrm{NX}_{k+1}$, we have VMax $(n) <$ VMax $(\mathrm{NX}_k - 1) <$ VMax $(\mathrm{NX}_{k+1} - 1)$

We assume that this result has been verified by calculation for all $n < 485$ (or even 10000), in order to be able to use the equivalence of the property V-3-2-1, either : For JGL(n), $r_n \approx 0.411 \times \dfrac{3^{m_n}}{2^n} d_n$

(In-1) $\Leftrightarrow \left(1 - \dfrac{3^{m_n}}{2^{n+1}}\right) v_0 \leq \dfrac{r_n}{2}$ with $\dfrac{3^{m_n}}{2^{n+1}} < 1$

$\Leftrightarrow$ VMax $(n) = \dfrac{r_n}{2} \times \dfrac{1}{1 - \dfrac{3^{m_n}}{2^{n+1}}} = 0.411 \times \dfrac{3^{m_n}}{2^{n+1}} d_n \times \dfrac{1}{1 - \dfrac{3^{m_n}}{2^{n+1}}}$, or more precisely, the integer part of this expression.

But, $\dfrac{3^{m_n}}{2^{n+1}} = \dfrac{3^{m_n}}{2^{m_n + d_n + 1}} = \mathrm{e}^{(d_n + 1)(\mathrm{Ln3} - \mathrm{Ln2})(m_n/(d_n + 1) - X)} = \mathrm{e}^{-u}$ with

$u = (d_n + 1)(\mathrm{Ln3} - \mathrm{Ln2})\left(X - \dfrac{m_n}{d_n + 1}\right)$

- Case where $\dfrac{m_n}{d_n + 1} - X$ is close enough to 0 to use $\mathrm{e}^{-u} \approx 1 - u$

  $$\mathrm{VMax}\,(n) = \frac{0.411 \times \dfrac{3^{m_n}}{2^{n+1}} d_n}{(d_n + 1)(\mathrm{Ln3} - \mathrm{Ln2})\left(X - \dfrac{m_n}{d_n + 1}\right)} \; \frac{0.411}{\mathrm{Ln3} - \mathrm{Ln2}} \times \frac{d_n}{d_n + 1} \times \frac{3^{m_n}}{2^{n+1}} \times \frac{1}{X - \dfrac{m_n}{d_n + 1}}$$

  As $d_n + 1 \approx d_n$

  $$\mathrm{VMax}\,(n) = 1.0136 \times \frac{3^{m_n}}{2^{n+1}} \times \frac{1}{X - \dfrac{m_n}{d_n + 1}}$$

  We notice that $\dfrac{3^{\mathrm{mX}_{k+1}}}{2^{\mathrm{NX}_{k+1}}} > \dfrac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}}$ and $X - \dfrac{\mathrm{mX}_{k+1}}{\mathrm{dX}_{k+1}} < X - \dfrac{\mathrm{mX}_k}{\mathrm{dX}_k}$ therefore

  VMax $(\mathrm{NX}_{k+1} - 1) >$ VMax $(\mathrm{NX}_k - 1)$

  For all $\mathrm{NX}_k < n < \mathrm{NX}_{k+1}$, $X - \dfrac{m_n}{d_n + 1} \geq X - \dfrac{\mathrm{mX}_k}{\mathrm{dX}_k}$, by definition

- If $X - \frac{m_n}{d_n+1} > X - \frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ then $\frac{3^{m_n}}{2^{n+1}} < \frac{3^{\mathrm{mX}_k}}{2^{\mathrm{NX}_k}}$ too and we indeed have $\mathrm{VMax}(n) < \mathrm{VMax}(\mathrm{NX}_k - 1)$
- If $X - \frac{m_n}{d_n+1} = X - \frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ then $n+1$ is a multiple of NX_k, i.e. there exists $j \in \mathbb{N}$, such as $n+1 = j\,\mathrm{NX}_k$, $m_n = j\,\mathrm{mX}_k$ and $d_n + 1 = j\,\mathrm{dX}_k$ and therefore $\frac{3^{m_n}}{2^{n+1}} = \left(\frac{3^{\mathrm{mX}_k}}{2^{\mathrm{dX}_k}}\right)^j$, but $\frac{3^{\mathrm{mX}_k}}{2^{\mathrm{dX}_k}} < 1$ therefore $\frac{3^{m_n}}{2^{n+1}} < \frac{3^{\mathrm{mX}_k}}{2^{\mathrm{dX}_k}}$ and finally $\mathrm{VMax}(n) < \mathrm{VMax}(\mathrm{NX}_k - 1)$, even in the best case where there is j pattern repetitions JGL(NX_k) to a few transitions permutations, which does not affect the calculation of r_n

- Case where $\frac{m_n}{d_n+1} - X$ is not close enough to 0 to use $\mathrm{e}^{-u} \approx 1 - u$

Then $u \geq b$ with $b = 0.1$, for example

because $\frac{\mathrm{e}^{-u} - (1-b)}{\mathrm{e}^{-b}} < \frac{\mathrm{e}^{-b} - (1-b)}{\mathrm{e}^{-b}} \approx 5.3 \times 10^{-3}$ is already a sufficient relative precision to apply the equivalence.

$u \geq b \Leftrightarrow -u \leq b \Leftrightarrow \mathrm{e}^{-u} \leq \mathrm{e}^{-b} \Leftrightarrow 1 - \mathrm{e}^{-u} \geq 1 - \mathrm{e}^{-b} > 0 \Leftrightarrow \frac{1}{1-\mathrm{e}^{-u}} \leq \frac{1}{1-\mathrm{e}^{-b}}$

$r_n \approx 0.411 \times \frac{3^{m_n}}{2^n} d_n \leq 0.411 \times 2\, d_n$ because $\frac{3^{m_n}}{2^{n+1}} < 1$

$\mathrm{VMax}(n) = \frac{r_n}{2} \times \frac{1}{1-\mathrm{e}^{-u}} \leq 0.411\, d_n \times \frac{1}{1-\mathrm{e}^{-u}} \leq \frac{0.411}{1-\mathrm{e}^{-b}} d_n = K\, d_n$, with $K = \frac{0.411}{1-\mathrm{e}^{-b}}$ therefore $\mathrm{VMax}(n)$ is of the order of d_n

For $b = 0.1$, $K \approx 4.31$

But for $\mathrm{NX}_k < 8573543875303$, $X - \frac{\mathrm{mX}_k}{\mathrm{dX}_k} < \frac{6}{\mathrm{dX}_k^{\,2}}$ (see IV-8, maximum reached for $= 12079$),

therefore $X - \frac{\mathrm{mX}_k}{\mathrm{dX}_k}$ is of the order of $\frac{1}{\mathrm{dX}_k^{\,2}}$ in (at least) the interval that interests us.

And $\mathrm{VMax}(\mathrm{NX}_k - 1) \approx 1.0136 \times \frac{1}{X - \frac{\mathrm{mX}_k}{\mathrm{dX}_k}}$ according to the property V-4-1

Therefore $\mathrm{VMax}(\mathrm{NX}_k - 1) > \frac{1.0136}{6} \mathrm{dX}_k^{\,2}$ is of the order of $\mathrm{dX}_k^{\,2}$ (for $\mathrm{NX}_k < 8573543875303$, what interests us here)

$\mathrm{VMax}(n) < \mathrm{VMax}(\mathrm{NX}_k - 1) \Leftrightarrow K\, d_n < \frac{1.0136}{6} \mathrm{dX}_k^{\,2} \Leftrightarrow d_n < 0.0392\, \mathrm{dX}_k^{\,2}$, which is always true because $d_n < \mathrm{dX}_{k+1}$ and dX_{k+1} satisfies this inequality for $\mathrm{dX}_k \geq 55$ and it's true with the true values (without an upper bound by 6) for $\mathrm{dX}_k \geq 10$ but anyway, it is assumed that the necessary condition has been verified for $n < 485$)

<u>Conclusion</u> : The necessary condition is indeed proven

Finally, the experimental results have been proven, we were also able to find an instantaneous method to determine JGL(n) and calculate $\mathrm{VMax}(n)$, which took a considerable amount of time (see the program in paragraph V-4).

6. **Proof of the experimental result of the paragraph IV-14 for $v_0 < 0$:**

We could be satisfied with the observation of the previous paragraph IV-14 regarding the rows of the table of results, but we seek to prove it here.

The goal is to prove that $\frac{m_n}{d_n+1}$ is a excess approximation of X (defined in paragraph X IV-9 with the Stern-Brocot tree), as well as some reducible fractions representing an approximation by excess, is a necessary and sufficient condition for the maximum value of v_0, called VMax(n), satisfying the inequality **(In-1)** increases.

We could provide an analogous demonstration to that of the previous paragraph.

We will focus on explaining the emergence of reducible fractions equal to excess approximations.

Indeed, the Stern-Brocot tree only gives irreducible fractions !

We saw that JGL had "patterns" of lengths equal to N_1 ("approximation of X by excess") which repeated themselves a number of times, noted p.

For $1 < k \leq p$ and $N_0 = k\,N_1$, we have $\frac{\text{VMax}\,(N_0-1)}{\text{VMax}\,(N_1-1)} = \frac{R_0}{R_1}$ since $X - \frac{m_0}{d_0} = X - \frac{m_1}{d_1}$ since the fractions are equal.

According to the formula of the paragraph V-3-1-d, we have $R_0 = \frac{1-(F_1)^k}{1-F_1} R_1$ for $1 < k \leq p$

But as $F_1 > 1$ because it is an approximation by excess,

then, for $1 < k \leq p$, $(F_1)^k > F_1 \Leftrightarrow 1-(F_1)^k < 1-F_1 \Leftrightarrow \frac{1-(F_1)^k}{1-F_1} > 1$

and finally $\frac{\text{VMax}\,(N_0-1)}{\text{VMax}\,(N_1-1)} > 1$, therefore the p lines should appear in the table.

If they do not appear it is related to truncation errors in the calculations. We can vary their emergence by manipulating the precision of the calculations.

Anyway, the value of $\text{VMax}\,(N_0-1) \approx \text{VMax}\,(N_1-1)$ therefore we can eliminate them.

Moreover, if there was a cycle of length N_1 then multiple lengths $k\,N_1$ would represent a non-minimum cycle.

Only the irreducible fractions are intertesting for significant increases of $\text{VMax}\,(n)$

VI. **Search for cycles :**

We have already demonstrated the importance of the list JGL(N) in the previous sections to determine the minimum length of a non-trivial cycle.

We will recall the results and establish a simple function to determine the minimum length of a cycle for v depending on the verification of the sequence.

Next, we will focus on the localization of the elements of a potential cycle of length N and we will get new and very interesting results.

We limit ourselves to the study of the cycles for which $\text{pgcd}(b_1, v_0) = 1$
Indeed, if we get a cycle with (b_1, v_0), then for any odd non-zero integer k, we also have a cycle with $(k\,b_1, k\,v_0)$

For $|b_1| < 1024$, the results are valid for $\mathrm{N} \geq 27$. This would be easily generalizable for larger values of b_1, with the minimum value of N increasing slowly.

We will prove that the elements of a non-trivial cycle of length N for v are in the interval $[\ 2^{0.04\mathrm{N}} \quad 2^{\mathrm{N}}\ [$ for $\mathrm{N} \geq 1000$ with $|b_1| < 1024$, by using :

- The "Theorem 0" to prove that the elements of a potential cycle are less than 2^{N} (for $v_0 > 0$ and, by symmetry for $v_0 < 0$, greater than -2^{N} for $\mathrm{N} \geq 27$ with $|b_1| < 1024$).
- A very robust statistical method (linear correlation coefficient greater than 0.999) to prove that the minimum element is greater than $2^{0.04\mathrm{N}}$ (for $v_0 > 0$ and, by symmetry for $v_0 < 0$, the maximum element is less than $-2^{0.04\mathrm{N}}$) for $\mathrm{N} > 1000$.

We can then conclude on the possible existence of other cycles.

Note: According to the proof of "Theorem 1", it is easy to show that revolutionary numbers (which are the solutions to "Theorem 1", from which we obtain almost a cycle), are the minimum solutions or not of the lists of transitions $\mathrm{L}(\mathrm{N}, m, d)$ for which $\frac{m}{d}$ is an approximation of X by defect or by excess.

1. **Summary on the minimum length of a non-trivial cycle :**

   1. **Recap of the results** :

      We have seen how the transition list test $\mathrm{JGL}(n)$ made it possible to know the minimum length N of a potential cycle for the reduced Syracuse sequence v.

      In the section IV, we found this length, simply by doing tedious calculations using a computer

      In the section V, we proved the experimental results by studying more precisely some properties of $\mathrm{JGL}(n)$ and this made it possible to obtain the result very easily.

      It turns out that this minimum length for a potential cycle depends on the maximum value v_0 (minimum for $v_0 < 0$) for which we know that the sequence has been tested.

      The length limit is increased for certain only if the verification has been made up to the value v_0 for N corresponding to the defect "approximations" of X (by excess for $v_0 < 0$), more precisely when $\mathrm{N} = m + d$ and $\frac{m}{d}$ is an approximation of X.

      With the tests for the standard Syracuse sequence, the result of Shalom Eliahou was found again, namely that the minimum length of a cycle is $\mathrm{N} = 114208327604$ for v, which represents a length of 186265759595 for u.

      Given the search for trivial cycles for $|v_0| < 2^{N_0 + B_1}$, with $N_0 = 17$ in the case $|b_1| < 1024$ (see paragraph III), we are assured :

      - For $v_0 > 0$, the minimum length of a cycle is 1539
      - For $v_0 < 0$ (or $b_1 < 0$), the minimum length of a cycle is 1054

      Maximum values of v_0 in the following tables are given for $b_1 = 1$ and are to be multiplied by b_1 if $b_1 \neq 1$.

      We find the non-exhaustive table below ($\mathrm{N}_u = \mathrm{N} + m$ is the minimum length for the Syracuse sequence u) :

      For $v_0 > 0$

N	N_u	max for v_0
27	44	$2^{6.6994}$
46	75	$2^{8.1032}$

65	106	$2^{9.7436}$
149	243	$2^{11.2271}$
233	380	$2^{12.2368}$
317	517	$2^{13.1703}$
401	654	$2^{14.253}$
485	791	$2^{16.1354}$
1539	2510	$2^{17.8632}$
2593	4229	$2^{18.6816}$
3647	5948	$2^{19.2426}$
4701	7667	$2^{19.6813}$
5755	9386	$2^{20.0495}$
6809	11105	$2^{20.3727}$
7863	12824	$2^{20.6657}$
8917	14543	$2^{20.9379}$
9971	16262	$2^{21.1959}$
11025	17981	$2^{21.4446}$
12079	19700	$2^{21.6882}$
13133	21419	$2^{21.93}$
14187	23138	$2^{22.1737}$
10439860591	17026679261	$2^{67.0843}$
114208327604	186265759595	$2^{71.4131}$
6048967074079039	9865440379489587	$2^{100.3901}$
4055842426088317893144813378287	6614794088504882375744109248036	$2^{200.3751}$

For $v_0 < 0$

N	N_u	max for v_0
3	5	$-2^{1.5850}$
11	18	$-2^{4.6439}$
19	31	$-2^{7.7279}$
84	137	$-2^{12.5758}$
569	928	$-2^{16.3061}$
1054	1719	$-2^{21.8053}$
25781	42047	$-2^{27.1956}$
50508	82375	$-2^{29.9750}$
766512153894657	1250127478260940	$-2^{99.7182}$
9881527843552324	16116077770794287	$-2^{107.8222}$

2. **Upper bound of f(N) with the maximum value of $v_0 \approx 2^{f(N)+B_1}$:**

   We set f(N), the exponent of the maximum value of $v_0=2^{f(N)+B_1}$ according to N.
   The local maximum of f(N) are obtained for the values of N corresponding to the defect "approximations" of X (by excess for $v_0 < 0$), therefore we can take these points to upper bound f(N)

But, in these cases of defect approximations of X, we can apply the property V-4-1 where we have :

$\max(v_0) \approx 1.0136\times\frac{b_1}{X-\frac{m}{d}} = \frac{1.0136\,b_1}{|diff|}$.

However, without using the best approximation theorem for continuous fractions which upper bounds the absolute difference, we can say that certainly $C < d^2|diff|$ with $C > 10^{-3}$, this lower bound can be modified without changing the reasoning.

We also have $d \approx \frac{\mathrm{N}}{1+X}$ for N large enough.

Therefore $\max(v_0) < \frac{1.0136\,d^2 b_1}{C} \approx \frac{1.0136\,\mathrm{N}^2 b_1}{(1+X)^2 C}$

And therefore $f(\mathrm{N}) < \frac{2\,\mathrm{Ln}(\mathrm{N})}{\mathrm{Ln}2} + \frac{\mathrm{Ln}(1.0136)-2\,\mathrm{Ln}(1+X)-\mathrm{Ln}(C)}{\mathrm{Ln}2} = \frac{2}{\mathrm{Ln}2}\mathrm{Ln}(\mathrm{N})+4$ with $C = 8.8198\times10^{-3}$, a change in the value of C does not change anything.

In the tests :

- this is the code of the paragraph V-4-1 with the displaying of additional columns in the results
- Maximum values of v_0 in the following tables are given for $b_1 = 1$ and are to be multiplied by b_1 if $b_1 \neq 1$.
- $\mathrm{N}_1 = 1054$
- *exp2* such as $v_0 \approx 2^{exp2}$
- $f(\mathrm{N}) = \frac{2}{\mathrm{Ln}2}\mathrm{Ln}(\mathrm{N})+4$
- diff_exp2, the difference between two consecutive values of *exp2*

For $v_0 > 0$:Test of the approximation of $f(\mathrm{N})$: Try

For $v_0 < 0$: Test of the approximation of $f(\mathrm{N})$: Try

3. **Upper bound of the difference between two consecutive values of *exp2* for N an "approximation of *X*"**

It is noted, in the previous test regarding the defect approximations
, but also in the previous test regarding the approximations by excess, that the difference between 2 consecutive values of *exp2* is always less than 10.

For the defect approximations :This difference is even less than 3 for $\mathrm{N} \leq 125743$

For the excess approximations : This difference is even less than 6 for $\mathrm{N} \leq 83130157078217$

This means that it only a few additional bits are needed for the maximum values of v_0 (up to the sequence is verified), to change level and have another minimum length value for a non-trivial cycle.

4. **Lower bound of the minimum length N of a cycle if the sequence has been verified up to $v_0 = 2^{n+B_1}$**

Conversely, if Syracuse sequence has been verified for $v_0 = 2^{n+B_1}$, then the minimum length of a potential cycle will be exponential.

And we solve $n < f(\mathrm{N})$

$\Leftrightarrow n < \frac{2}{\mathrm{Ln}2}\mathrm{Ln}(\mathrm{N})+4$

$\Leftrightarrow \mathrm{Ln}(\mathrm{N}) > \frac{n-4}{2}\mathrm{Ln}2 = \frac{n}{2}\mathrm{Ln}2 - 2\,\mathrm{Ln}2$

$\Leftrightarrow \mathrm{N} > \frac{2^{n/2}}{4}$

For $n = 68$, we find $N = \dfrac{17179869184}{4} = 4294967296 < 10439860591$, which is consistent with the previous table, the lower bound of C is large.

This relatively large lower bound is obtained in the general case and could be finer for specific values of n

Property VI-1-4 : If Syracuse sequence is verified up to $v_0 = 2^{n+B_1}$ (with $n \geq N_0$), then the minimum length of a cycle (not trivial) is $N > \dfrac{2^{n/2}}{4}$, for the reduced Syracuse sequence

2. **"Theorem 0"**

Statement :

For any list of transitions L (of any length N), there exists an infinite number of roots (values of v_0) whose first transitions with the reduced sequence of Syracuse v are those of the given list L.
There exists one and only one root s_0 in the interval $[0, 2^N[$ and the other roots are of the form $V_0 = s_0 + n \times 2^N$ with n > 0

I) Notations and definitions :

First, let's define some elements and specify some notations

a) Trajectory of length $N = m + d$ for v:

A trajectory of length N is a list of "type 0" or "type 1" transitions, therefore a word composed of 0 and 1

We will set d, the number of "type 0" transitions and m the number of "type 1" transitions and $N = m + d$:

For example :

With v, we have $v_0 = 7$ =1=> 11 =1=> 17 =1=> 26 =0=> 13, the trajectory is "1110"

Instead of considering a trajectory as a word on the alphabet {0, 1}, it can be seen as a number in binary format, written with the least significant bits on the left.

For example "1110" translates to $1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 = 7$.

Definition : A trajectory for v is named $T_N = \sum_{i=0}^{N-1} t_i \times 2^i$ where t_i is the type of transition i.

It is a number strictly less than 2^N

b) Extension of v for $v_0 = 0$

This extension, to standardize, simply allows having $v_0 = 0$ as a solution of $T_N = 0$ for any value of $N > 0$ instead of 2^N and therefore simplify the reasoning.
It's not about having another trivial cycle of length 1 starting from 0.

II) Proof :

Another proof, different from this one, written with another approach is also available.

1) Let us prove that there is one and only one solution v_0 in $[0, 2^N[$ such that v_0 follows a given trajectory T_N .

That is to say that there exists a bijection between the trajectories and the solutions which follow them in the set $[0, 2^N[$.
Main ideas of the proof :
- the parity of a number is a local property that only depends on the least significant bit (bit 0) and therefore 2^n is even for n > 0.

- the transitions t_i for i < n only depend on the (n-1) least significant bits of v_0 or that the transition t_n depends on the result of the previous transitions on the $n-1$ least significant bits of v_0 and on the bit n of v_0, this is the new idea about the Syracuse sequence.

We prove this result by induction.

a) Let's verify that the property is true for $N = 1$:
For N = 1, the only two trajectories of length 1 are "0" or "1" for t_0 of "type 0" or "type 1"
Thanks to the extension of v for $v_0 = 0$, v_0 satisfies "0" (in this particular case, we could have taken $v_0 = 2^{\mathrm{N}} = 2$ and define a bijection between [0;2^{N}-1] and [1;2^{N}])
And $v_0 = 1 < 2^1$ satisfies "1" because 1 is odd.
So the property is true for N = 1
b) Let's prove that if the property is true for N, then it is true for N + 1
We consider $T_{\mathrm{N}+1}$, a trajectory of length N + 1 or list of transitions of the sequence v with N + 1 elements.
A solution which follows $T_{\mathrm{N}+1}$ must necessarily follow T_{N}, the sub-trajectory of the first N transitions of $T_{\mathrm{N}+1}$, this is a necessary condition.
Since the property is true for N, then there exists one and only one $0 \le s_0 < 2^{\mathrm{N}}$ following T_{N}.
Let's take $v_0(a) = a \times 2^{\mathrm{N}} + s_0$ with a ∈ {0, 1} so $v_0(a) < 2^{N+1}$
The term $a \times 2^{\mathrm{N}} = 0$ if $a = 0$ or $a \times 2^{\mathrm{N}} = 2^{\mathrm{N}}$ if $a = 1$ and this term therefore remains even for all the first N transitions of T_{N}, since we divide by 2 at each transition (even if we have a "type 1" transition)

Let's detail the first transition :

$v_0(a)$ has the parity of s_0 linked to $t_0 \in \{0, 1\}$

If $t_0 = 0$ then s_0 is even (because s_0 follows T_{N}) and $v_1(a) = \dfrac{v_0(a)}{2} = \dfrac{a \times 2^{\mathrm{N}} + s_0}{2} = a \times 2^{\mathrm{N}-1} + \dfrac{s_0}{2} = a \times 2^{\mathrm{N}-1} + s_1$
If $t_0 = 1$ then s_0 is odd (because s_0 follows T_{N}) and
$v_1(a) = \dfrac{3v_0(a)+1}{2} = \dfrac{3(a \times 2^{\mathrm{N}} + s_0)+1}{2} = a \times 3 \times 2^{\mathrm{N}-1} + \dfrac{3s_0+1}{2} = a \times 3 \times 2^{\mathrm{N}-1} + s_1$
$v_1(a)$ has the parity of s_1 which follows t_1...

We can easily prove by induction that :

For all $0 \le n \le \mathrm{N}$, $v_n(a) = a \times 3^{m_n} \times 2^{N-n} + s_n$ with m_n, the number of "type 1" transitions in the sub-trajectory of the first n transitions of T_{N}
We have verified this property for $n = 0$ (with $m_0 = 0$) and for $n = 1$ and the rest of the reasoning for this recurrence is also very simple.

By reiterating the reasoning, after N transitions, or by using the recurrence formula, we have :

If m is the number of "type 1" transitions of T_{N} then

$v_{\mathrm{N}}(a) = a \times 3^m + s_{\mathrm{N}}$, this is the key of the reasoning.

Two cases are possible for the transition N + 1

1. If $t_{\mathrm{N}} = 0$ then $v_0(a)$ follows the trajectory $T_{\mathrm{N}+1}$ if and only if $v_{\mathrm{N}}(a)$ is even:
   - If s_{N} is even then $v_N(a)$ is even if and only if $a = 0$ and $v_0(0)$ follows $T_{\mathrm{N}+1}$
   - If s_{N} is odd then $v_N(a)$ is even if and only if $a = 1$ (because 3^m is odd) and $v_0(1)$ follows $T_{\mathrm{N}+1}$
2. If $t_{\mathrm{N}} = 1$ then $v_0(a)$ follows the trajectory $T_{\mathrm{N}+1}$ if and only if $v_{\mathrm{N}}(a)$ is odd:
   - If s_{N} is odd then $v_N(a)$ is odd if and only if $a = 0$ and $v_0(0)$ follows $T_{\mathrm{N}+1}$
   - If s_{N} is even then $v_N(a)$ is odd if and only if $a = 1$ (because 3^m is odd) and $v_0(1)$ follows $T_{\mathrm{N}+1}$

Conclusion: The property is true for $N+1$, which completes the reasoning by induction.

2) Infinity of solutions that follow a trajectory T_N
To prove that there are infinitely many solutions in $\mathbb{N}$ just consider the values:

$V_0 = s_0 + n\times 2^N$ with $n \in \mathbb{N}$ where s_0 is the solution that follows T_N with $s_0 < 2^N$ according to 1)

because, by construction, V_0 follows T_N and $V_N = s_N + n\times 3^m$

The proof of this theorem is done in the standard case but it is generalized without any difficulty for any b_1.
We can even generalize this theorem by taking any positive odd coefficient a_3 instead of the restrictive case 3 by considering :

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \dfrac{a_3 v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases} \quad \text{with } a_3 = 2a+1 \text{ and } a \geq 0$$

End of the proof of Theorem 0

Only for the following tests : a_3 = 3 , b_1 = 1
Search for minimum v_0 for the list : 10001001110100111011 Try
Search for minimum v_0 for a random list of maximum length : 50 Random test See/Hide the code

3. **Location of elements of potential cycle using the "Theorem 0"**

We set V_0 the minimum solution in absolute value which follows the transition list $L(N, m, d)$ for b_1

Here, we take the extension with $b_1 < 0$ instead of $v_0 < 0$

According to the "Theorem 0", we know that $V_0 \in]\,0 \quad 2^N\,[$ and that all the values of v_0 which have as first transitions those of $L(N, m, d)$ are of the form : $v_0 = V_0 + a\times 2^N$ with $a \geq 0$ and $v_N = V_N + a\times 3^m$
Therefore $v_N - v_0 = V_N - V_0 + a(3^m - 2^N)$

We limit ourselves to the cycles for which b_1 and V_0 are coprimes i.e. $\gcd(b_1, V_0) = 1$.

If we have a cycle from v_0 then $v_N = v_0$

Therefore $v_N - v_0 = 0 \Leftrightarrow V_N - V_0 = a(2^N - 3^m) \Leftrightarrow a = \dfrac{V_N - V_0}{2^N - 3^m}$ for $N > 1$. The case $N = 1$ is trivial.

- If $V_N - V_0 = 0$ then $a = 0$ which means that we have a cycle from V_0
- If $V_N - V_0 \neq 0$ then $a > 0$ because we optionally take the option $b_1 < 0$ and $v_0 > 0$. Of course, it is necessary that $a \in \mathbb{N}$.

  Moreover, it is necessary that $v_0 = V_0 + a\times 2^N < \text{VMax}(N-1) < 2^{f(N)+B_1}$
  Therefore that $a\times 2^N < \text{VMax}(N-1)$, while maintaining $V_0 > 0$.

  According to the results of IV-13 which gives more accurate results for small values of N that in the previous paragraph (paragraph IV-14 for $b_1 < 0$)
  As a reminder with $|b_1| = 1$ and $\text{VMax}(N-1)$ is to be multiplied by $|b_1|$:

N	VMax (N−1)

8	$24.54 \sim 2^{4.617}$
27	$108.01 \sim 2^{6.755}$

and for $b_1 < 0$

N	VMax (N − 1)
11	$27.10 \sim 2^{4.760}$
19	$219.31 \sim 2^{7.777}$

In the standard case ($b_1 = 1$), we always have $\text{VMax}(\text{N}-1) < 2^{\text{N}}$ therefore the case $a \neq 0$ is not possible.

As for $|b_1| < 1024$, we have $B_1 \leq 10$.

For $\text{N} \geq 27$, we always have $\text{VMax}(\text{N}-1) < 2^{\text{N}}$ then the case $a \neq 0$ is not possible.

For $\text{N} < 27$ and $b_1 > 0$ or for $\text{N} < 19$ and $b_1 < 0$, it may be that in some cases, we can have solutions for a cycle without it being the minimum solution V_0.

We can revisit the results obtained by looking at when the maximum value of the cycle is greater than 2^{N} to see that we are always in this case $V_0 + a \times 2^{\text{N}} = V_0 + \dfrac{V_{\text{N}} - V_0}{2^{\text{N}} - 3^m} \times 2^{\text{N}}$ where V_0 is the minimum solution for the list $\text{L}(\text{N}, m, d)$ with this maximum value as the first value of the cycle.

It can be noted that in even rarer cases, this is the case even for the minimum value of the cycle and therefore all the elements of the cycle are of this form.

The table gives all the cases where the value is not the minimum solution for the transition list.

Structure of the table :

<table>
<tr><td rowspan="2">b_1</td><td rowspan="2">v_0 (minimum value of cycle)</td><td rowspan="2">N</td><td colspan="6">List
List of values of the cycle
$2^{\text{N}} - 3^m$</td></tr>
<tr><td>v_0 (cycle value greater than 2^{N})</td><td>L</td><td>V_0</td><td>V_{N}</td><td>a</td><td>v_n (must be equal to this v_0)</td></tr>
</table>

For $b_1 > 0$ See/Hide results

Results at the end of the document

See the results

For $b_1 < 0$ See/Hide results

Results at the end of the document

See the results

If we have a cycle for $\text{L}(\text{N}, m, d)$, then the only solution is the minimum solution V_0 which follows the list of transitions , in the case where $\text{N} \geq 27$ with $|b_1| < 1024$).

Taking the N circular permutations on this list, we have the following property :

Property VI-3 : If v has a cycle of length N, then all the different elements of the cycle $\{v_0, ..., v_{\text{N}-1}\}$ satisfy $v_n < 2^{\text{N}}$ for $0 \leq n < \text{N}$ (for $\text{N} \geq 27$ if $|b_1| < 1024$)

4. **Probabilistic reasoning with the fundamental property :**

1. **Determination of the number Cy (N) of possible candidates v_0 for a cycle of length N :**

   1. **Special case : N is an "approximation of X"**

      For $v_0 > 0$, N corresponds to a defect "approximation".

      For $v_0 < 0$, N corresponds to an "approximation" by excess.

      According to the location of the values of a cycle of length N (previous paragraph VI-3), all elements v_0 are less than 2^N and are also the minimum solutions of a list L(N, m, d) for $N \geq 27$ and $|b_1| < 1024$.

      Counting the number of possible v_0 amounts to counting the number of lists L(N, m, d) with the bijection of "Theorem 0".

      The number of lists L(N, m, d) is easy to count, it is the number of ways to place m values "1" (or d values "0") in N positions, which corresponds to the number of combinations $C_N^m = \frac{N!}{m!(N-m)!}$

      with $n! = \prod_{i=1}^{n} i$, so the factorial of n.

      If we take into account that cycles would be equivalent and can be identified by their smallest element (or their largest element if $v_0 < 0$).

      If we set Cy (N) the number of v_0 possible candidates, v_0 minimum (v_0 maximum if $v_0 < 0$) for the cycle they represent, we have :

      $$\mathbf{Cy\,(N) = \frac{C_N^m}{N}}$$

   2. **General case : Any N**

      For $v_0 > 0$: But we can easily count the exact number of possible candidates (or number of transtion lists) thanks to the JGL(N) boundary list (described in IV and V) which is the limit list which ensures that for all $0 < n \leq N$, $v_n \geq v_0$ and v_n minimum since "type 0" transitions are prioritized, among all lists verifying for all $0 < n \leq N$, $v_n \geq v_0$
      Symmetrically, for $v_0 < 0$: we can easily count exactly the number of potential transtion lists thanks to the JGL(N) boundary list which is the limit list which ensures that for all $0 < n \leq N$, $v_n \leq v_0 < 0$ and v_n maximum since we prioritize "type 0" transitions, among all the lists verifying for all $0 < n \leq N$, $v_n \leq v_0 < 0$

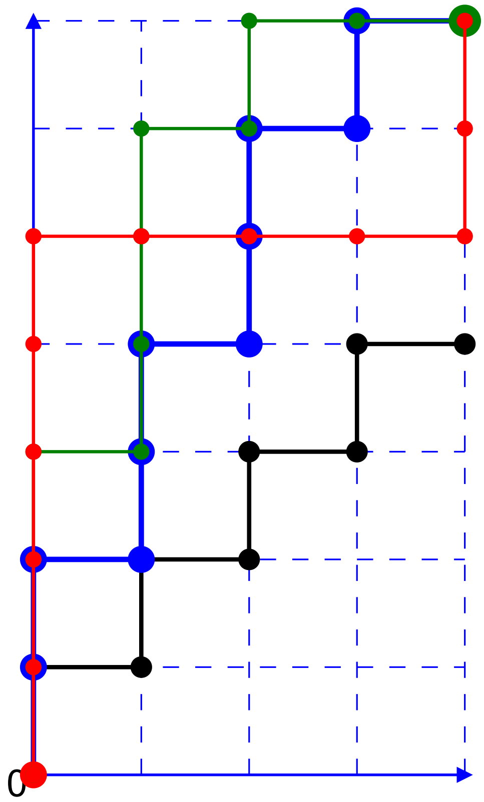
      In fact, the possible candidates are the solutions of the lists L(N, m, d) for which at each length $0 < i \leq N$, the number of "type 1" transitions for the i first transitions in the L(N, m, d) list is greater than or equal to that for the JGL(N). Moreover m must be equal to m_N, the number of "type 1" transitions of JGL(N).

      For $v_0 > 0$: For values of $\frac{m}{d} < X$ corresponding to potential cycles (to better approximations of X by defect), we modify the last transition of JGL(N) by shifting it to "type 0".

      We can create a diagram by representing the number of "type 1" transitions on the vertical axis. Transition lists are then paths with movements upward or to the right (toward the North or toward the East). The valid transition lists are those that remain above the JGL boundary and end after N steps at the same point as JGL(N).

      - The blue line represents the boundary JGL

- The green line corresponds to a valid transition list because it is always above JGL
- The red line corresponds to a list of transitions that is not valid because it passes under JGL
- The black line corresponds to the boundary for "words of Dyck" (or Catalan's triangle)
- The large green dot is the ending point of the valid transition lists (at the same location as the one for JGL(N))



Algorithms for counting solution lists (they can also be used to generate them exhaustively) :

- Recursive method : We construct (or simply count) the list to the depth N maintaining its length and the number m of "type 1" transitions.
  - If, for the length n, m is greater than or equal to the number of transitions of "type 1" of JGL(n), then we can add a transition of "type 0"
  - If, for the length n, m is strictly less than the number `nmax` of "type 1" transitions of JGL(N), then we can add a transition of "type 1"
- Method with memorization of the number of lists $\mathrm{Nb}(n, m)$ for a length n and a given number of "type 1" transitions m.

  We have :
  - $\mathrm{Nb}(0, 0) = 1$
  - $\mathrm{Nb}(0, k) = 0$ for $1 \leq k \leq \mathrm{N}$
  - The elements are constructed for $n+1$ from those of n as follows :
  - $\mathrm{Nb}(n+1, k) = 0$ for $0 \leq k \leq \mathrm{N}$
  - $\mathrm{Nb}(n+1, k)$ += $\mathrm{Nb}(n, k-1)$ for $0 < k \leq m_{\mathrm{N}}$ (possible addition of a transition of "type 1")
  - $\mathrm{Nb}(n+1, k)$ += $\mathrm{Nb}(n, k)$ for $k \geq m_{n+1}$ (possible addition of a transition of "type 0")
  - The number of lists we are interested in is $\mathrm{Nb}(\mathrm{N}, m_{\mathrm{N}})$

  This method is much faster.

This reminds the words of Dyck or Catalan's triangle[5][6] with m times the letter Y (y-axis of the coordinates) and d times the letter X (x-axis of the abscissas) to retain only the paths "above" of the bisector, with always more letters y than x in the "sub-words"

We use the formula proven in the English Wikipedia page about Catalan's triangle[6] for n Y and k X :

$C(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1} = \binom{n+k}{n} - \binom{n+k}{n+1}$

By replacing $n+k$ by N, n by m_N and k by d_N

$C(m_N,d_N) = \binom{N}{m_N} - \binom{N}{m_N+1} = \frac{2m_N - N+1}{m_N+1}\binom{N}{m_N} = \frac{m_N - d_N+1}{m_N+1}\binom{N}{m_N} = \left(1-\frac{d_N}{m_N+1}\right)\binom{N}{m_N}$

As the JGL boundary is above the first bisector, because $m > d$, it is therefore more selective for the possible paths,

We have the immediate upper bound :

$$\boldsymbol{Cy\ (N) < \left(1-\frac{d_N}{m_N+1}\right)\binom{N}{m_N} = \left(1-\frac{d_N}{m_N+1}\right)C_N^{m_N}}$$

3. **Upper bound of *Cy* (N) for large values of N**

As the minimum length of a cycle is at least 1000, it is wise to have an upper bound in this case.

When N is large, $m_N+1 \approx m_N$, $m_N \approx \frac{X}{1+X}N$ and $d_N \approx \frac{1}{1+X}N$ and therefore :

$1-\frac{d_N}{m_N+1} \approx \frac{X-1}{X} \approx 0.415 < 1$

We will therefore take $Cy\ (N) < C_N^{m_N}$, already with an upper bound of a factor of at least 2

In paragraph IV-16, we saw that $m_N = \lceil \frac{Ln2}{Ln3} N \rceil$

Then $Cy\ (N) < C_N^{\lceil N\,Ln2/Ln3 \rceil}$ for N > 1000

Let's look for an equivalent of $C_N^{\lceil N\,Ln2/Ln3 \rceil}$ when N large enough :

$Ln(n!) = \sum_{i=1}^{n} Ln(i) \approx \int_1^n Ln(X)dX = \left[X Ln(X) - X\right]_1^n = n\,Ln(n) - n + 1$

We can use the Stirling formula : $Ln(n!) \approx n\,Ln(n) - n$ for n large enough ($n \geq 100$ for a relative error of 10^{-2}).

As $C_n^p = \frac{n!}{p!(n-p)!}$, we have :

$Ln(C_n^p) = Ln\left(\frac{n!}{p!(n-p)!}\right) = Ln(n!) - Ln(p!) - Ln((n-p)!)$

$Ln(C_n^p) \approx n\,Ln(n) - n - p\,Ln(p) + p - (n-p)Ln(n-p) + n - p$

$Ln(C_n^p) \approx n\,Ln(n) - p\,Ln(p) - (n-p)Ln(n-p)$ after simplifications

As $\lceil \frac{N\,Ln2}{Ln3} \rceil > \frac{Ln2}{Ln3} N > \frac{N}{2}$ for N large enough, and $\frac{Ln2}{Ln3} = \frac{X}{1+X}$ with $X = \frac{Ln2}{Ln3 - Ln2} \approx$ 1,70951129135

$Ln(C_N^{\lceil N\,Ln2/Ln3 \rceil}) \approx N\,Ln(N) - \frac{XN}{1+X} Ln\left(\frac{XN}{1+X}\right) - \frac{N}{1+X} Ln\left(\frac{N}{1+X}\right)$

$Ln(C_N^{\lceil N\,Ln2/Ln3 \rceil}) \approx$

$N\,Ln(N) - \frac{XN}{1+X}Ln(X) - \frac{XN}{1+X}Ln(N) + \frac{XN}{1+X}Ln(1+X) - \frac{N}{1+X}Ln(N) + \frac{N}{1+X}Ln(1+X)$

$Ln(C_N^{\lceil N\,Ln2/Ln3 \rceil}) \approx \frac{-XN}{1+X}Ln(X) + \frac{XN}{1+X}Ln(1+X) + \frac{N}{1+X}Ln(1+X)$ after simplifications

$$\mathrm{Ln}\,(C_N^{[N\,\mathrm{Ln}2/\mathrm{Ln}3]}) \approx \left(\mathrm{Ln}\,(1+X) - \frac{X}{1+X}\mathrm{Ln}\,(X)\right)\mathrm{N}$$

$$\mathrm{Ln}\,(C_N^{[N\,\mathrm{Ln}2/\mathrm{Ln}3]}) \approx 0.658459\ \mathrm{N}$$

$$C_N^{[N\,\mathrm{Ln}2/\mathrm{Ln}3]} \approx e^{0.658459\,\mathrm{N}}$$

To get the number of bits of $\mathrm{Ln}\,(C_N^{[N\,\mathrm{Ln}2/\mathrm{Ln}3]})$, we divide by Ln2

We obtain a number of bits close to $\frac{0.658459}{\mathrm{Ln}2}\mathrm{N} \approx 0.949956\ \mathrm{N} < 0.95\,\mathrm{N}$

Therefore $C_N^{m_N} \approx 0.6\times 2^{0.95\,\mathrm{N}} < 2^{0.95\,\mathrm{N}}$

or $C_N^{m_N} \approx 10^{0.286\,\mathrm{N}}$

Then, Cy (N) < $C_N^{m_N} < 2^{0.95\mathrm{N}}$, the upper bound is already very large (639 times greater for N = 1539 , 438 times greater for N = 1054 in case $v_0 < 0$ than the number of Dyck words).

We will take this approximate value by excess :

Cy (N) < $2^{0.95\,\mathrm{N}}$ for N > 1000.

<u>Remarks</u> :

- Taking this large upper bound, it is as if we are making no restriction on the paths and taking the C_N^m transition lists and therefore that we did not consider v_0 minimum (v_0 maximum in case $v_0 < 0$)
- In the case that N corresponds to an "approximation" of X, then Cy (N) = $\frac{C_N^{m_N}}{\mathrm{N}}$ and therefore Cy (N) < $2^{0.95\,\mathrm{N} - \mathrm{N}/\mathrm{Ln}2}$

4. **Test**

Test for the length N : 65 [Try] [See/Hide code]

For N = 65, clicking on the "Try" button, we get the line : "Number of lists for cycle : ~ $2^{52.44}$ ~ $2^{0.81\times \mathrm{N}}$ = 6113392816333320"

Then we get the following results (even with a few couples $\frac{m}{d} > X$, marked with *) :

N	Number of transition lists with $v_n > v_0$
19*	~ $2^{11.37}$ ~ $2^{0.60\times \mathrm{N}}$ = 2652
27	~ $2^{18.25}$ ~ $2^{0.68\times \mathrm{N}}$ = 312455
46	~ $2^{35.15}$ ~ $2^{0.76\times \mathrm{N}}$ = 38036848410
65	~ $2^{52.44}$ ~ $2^{0.81\times \mathrm{N}}$ = 6113392816333320
84*	~ $2^{69.93}$ ~ $2^{0.83\times \mathrm{N}}$ = 1122428422670255691408
317	~ $2^{88.40}$ ~ $2^{0.91\times \mathrm{N}}$ = 65680667541548409420010089823370922159680503331172842091752
401	~ $2^{367.69}$ ~ $2^{0.92\times \mathrm{N}}$ = 484191448994484140914487401835997466375231964300210692319
485	~ $2^{447.07}$ ~ $2^{0.92\times \mathrm{N}}$ = 38176868759050134271911329690220347628482378391118516565097049761310997707
1054*	~ $2^{985.92}$ ~ $2^{0.94\times \mathrm{N}}$
1539	~ $2^{1445.83}$ ~ $2^{0.94\times \mathrm{N}}$
14187	~ $2^{13456.06}$
25187	~ $2^{24468.55}$

We can verify that for these different values of N, we have $Cy\,(\mathrm{N}) = \frac{\mathrm{C}_{\mathrm{N}}^{m}}{\mathrm{N}}$

Remarks :

- In the case of the standard Syracuse sequence, we found that the minimum length of a cycle is $\mathrm{N} = 114208327604$.

  Generating a list randomly is possible (we can even construct it step by step), testing this single list, ie, check if $v_{\mathrm{N}} = v_0$, is already out of reach with my personal computer, but we would have to test $Cy\,(\mathrm{N}) \approx 10^{0.286 \times 114208327604} \approx 10^{32663581695}$ lists which is unrealistic. Furthermore, we would then need to start again for all the possible values of N... this is certainly not the right method of reasoning.
- If we take as a value of N, a value that corresponds to an "approximation" of X by defect or by excess, $\mathrm{N} \times Cy\,(\mathrm{N})$ represents the number of revolutionary numbers. Thus, we can distinguish between two notions, rarity and abundance, which are not antinomical.

2. **Property VI-4-2 : The probability of "v_n even" is $\frac{1}{2}$**

This property is commonly accepted but let's take a look in the standard case.

It is only to observe the "shuffling" of the bits for potential encryption applications.

For $b_1 \neq 1$, we will accept this property.

- If $b_1 = 2b+1 > 0$ and $v_0 > 0$, then it is enough to consider the decomposition in base 2 of b to get the similar result.
- If $b_1 = 2b+1 < 0$, then :
  - If $v_0 < -b_1$, then there exists n such as $v_n < 0$ and we can reduce it to the previous case by changing v_n and b_1 in their opposites
  - If $v_0 > -b_1$, then $\frac{3v_n + b_1}{2} > v_n$ and we should consider the subtraction...

In the property, we can add the equiprobability for each bit to be equal to 0 or 1.

This is true when we are in the reduced trivial cycle $\{1,2\}$, we consider the case $v_0 > 4$.

Let's do a reasoning by induction.

It's true for v_0, since v_0 is arbitrary.

We assume that the property is true at rank n and we prove that this is true at rank $(n+1)$.

Let $v_n = \sum_{p=0}^{\mathrm{N}} a_p \times 2^p$ with $a_p = 0$ or $a_p = 1$, its decomposition in base 2.

- If $a_0 = 0$ then v_n is even and $v_{n+1} = \sum_{p=1}^{\mathrm{N}} a_p \times 2^{p-1} = \sum_{p=0}^{\mathrm{N}-1} a'_p \times 2^p$ with $a'_p = a_{p+1}$ therefore v_{n+1} has the parity of $a'_0 = a_1$ which is equiprobably 0 or 1. For the other bits, the equiprobability is direct.
- If $a_0 = 1$ then v_n is odd and

$$v_{n+1} = \frac{3v_n + 1}{2} = \frac{1 + v_n + 2v_n}{2} = \frac{1 + \sum_{p=0}^{N} a_p \times 2^p + \sum_{p=0}^{N} a_p \times 2^{p+1}}{2} = \sum_{p=0}^{N+1} a'_p \times 2^p \text{ with}$$

$$\begin{cases} r_0 = 1 \\ n_p = r_p + a_p + a_{p+1} \\ a'_p = n_p \text{ modulo } 2 \\ r_{p+1} = 1 \text{ if } n_p \geq 2 \\ r_{p+1} = 0 \text{ if } n_p < 2 \end{cases}$$

Therefore v_{n+1} has the parity of
$a'_0 = n_0 \text{ modulo } 2 = (r_0 + a_0 + a_1) \text{ modulo } 2 = (1+1+a_1) \text{ modulo } 2 = a_1$ which is equiprobably 0 or 1.

For the other bits :

- If $r_p = 0$ then $a'_p = (0 + a_p + a_{p+1}) \text{ modulo } 2$ is equiprobably 0 or 1.
- If $r_p = 1$ then $a'_p = (1 + a_p + a_{p+1}) \text{ modulo } 2$ is equiprobably 0 or 1.

This proves the property at rank $(n+1)$ and therefore, in particular that the probability of "v_n even" is $\frac{1}{2}$

3. **Distribution of the values of v_0, solutions of candidate lists**

Here, v_0 represents the minimum solution for a candidate transition list

1. **The probability that $v_0 < 2^{N-k}$ is $\frac{1}{2^k}$**

Notations :

- $v_0 = v_{0,N} = \sum_{n=0}^{N-1} d_n \times 2^n < 2^N$ is the minimum solution for the L(N, m, d)
(for $N \geq 27$ and $|b_1| < 1024$, this is the only solution that allows you to have a cycle)
- $v_{0,N-k} < 2^{N-k}$ is the minimum solution for the $(N-k)$ first transitions of a list L(N, m, d)
- $v_{n,N-k}$ is the current value obtained after n steps from the minimum solution $v_{0,N-k}$
- p is the probability that the last transition t_{N-1} of the L(N, m, d) list is of "type 1"

We accept that the value $v_{N-1,N-1}$ (the current value after $N-1$ steps from $v_{0,N-1}$) has an equiprobable parity ($\frac{1}{2}$ to be even, $\frac{1}{2}$ to be odd)

If (($v_{N-1,N-1}$ is even and t_{N-1} is of "type 1") or ($v_{N-1,N-1}$ is odd and t_{N-1} is of "type 0")) whose probability is $\frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$, which is independent of p,

then the last transition is not natural ($d_{N-1} \neq 0$), i.e. t_{N-1} does not match the type of transition obtained from $v_{N-1,N-1}$ and then the minimum solution is $v_{0,N} = v_{0,N-1} + 2^{N-1}$ (according to VI-2) and $v_{0,N} \geq 2^{N-1}$

Conversely, if (($v_{N-1,N-1}$ is even and t_{N-1} is of "type 0") or ($v_{N-1,N-1}$ is odd and t_{N-1} is of "type 1")) whose probability is $\frac{1}{2}(1-p) + \frac{1}{2}p = \frac{1}{2}$, which is independent of p,

then the last transition is natural, i.e. t_{N-1} corresponds to the type of transition obtained from $v_{N-1,N-1}$ and then the minimum solution is $v_{0,N} = v_{0,N-1}$ (according to VI-2) and $v_{0,N} < 2^{N-1}$
So the probability that $2^{N-1} \leq v_{0,N} < 2^N$ is equal to $\frac{1}{2}$

Repeating the same reasoning for the k last transitions,

- The probability that $2^{N-k} \leq v_{0,N-k} < 2^{N-k+1}$ is equal to $\frac{1}{2^k}$
- $v_{0,N} = v_{0,N-k} < 2^{N-k}$ if and only if the k last transitions are natural from $v_{0,N-k}$ and the probability is $\frac{1}{2^k}$

- Conversely, the probability that $2^{N-k} \leq v_{0,N}$ is $1 - \dfrac{1}{2^k}$

It remains to determine for which values of k this relation is valid.

2. **Values of v_0 which loop in a trivial cycle for a list**

The previous reasoning would not hold for the values of v_0 that loop in a cycle for a candidate list of transitions.

It is therefore important to consider these potential cases.

In the standard case, for any candidate transition list, as we have $v_n > v_0$ for all $0 < n < N$ and that the JGL list begins with the "11" transitions (to avoid the trivial cycle and therefore $v_0 > 1$), we are assured that the value $v_n = 1$ is never reached and therefore we are not in the trivial cycle even if the list of transitions ends with a large number of alternations "10"

For $v_0 > 0$, by construction, the JGL list already eliminates the values of v_0 which are the minimum values of the trivial cycles since it is necessary to "modify" the last transition into transition of "type 0" to have a cycle because we always consider for N, the minimum length of a cycle, which corresponds to an "approximation" of X.

For $v_0 < 0$, if the minimum values of a trivial cycle form a cycle too, this would mean that N is a multiple of this length, which is not interesting. But it is mainly impossible because it would take exactly the same proportion of "type 1" transitions (for example) and the Stern-Brocot algorithm gives irreducible fractions.

It remains to identify the lists for which we can reach a trivial cycle from an initial value lower than the minimum value of the cycle.

This phenomenon must be quite rare compared to the number of lists Cy (N), but we will study it more precisely in the study of the divergence (paragraph VII-2) by revisiting the results obtained for $|b_1| < 1024$
But for these values, the proportion of transitions should be identical to that of JGL and then, at the end, one or several transitions allow you to return to the initial (strictly lower) value, which would break the cycle. This break is equivalent to at least one high-order bit in v_0 and therefore v_0 can not be less than the minimum value of the cycle.

Conclusion : Such lists are not candidate lists to have a cycle of length N, the minimum possible length, which necessarily corresponds to N "approximation" of X

4. **Upper bound of the number of $v_0 < 2^{N-k}$**

As Cy (N) is very large and even as long as $\dfrac{Cy\ (N)}{2^k}$ is large, then we must approach a distribution of v_0 close to the probabilities. Even if the Syracuse sequence is perfectly determined from v_0 and it is therefore not a random phenomenon, but very deterministic and mechanical, it is interesting to see it as a random phenomenon, which follows a probability distribution.

What seems chaotic to us for a value of v_0 (or a transition list) is no longer at all when looking at the whole.

In first approximation, to get an idea, by looking at what happens from the last transitions, we can say that the v_0 are distributed as follows :

$\dfrac{Cy\ (N)}{2}$ are in [2^{N-1} $\quad 2^N$ [

$\dfrac{Cy\ (N)}{4}$ are in [2^{N-2} $\quad 2^{N-1}$ [

...

$\frac{Cy\,(\mathrm{N})}{2^k}$ are in [$2^{\mathrm{N}-k}$ $2^{\mathrm{N}-k+1}$ [

and therefore $\frac{Cy\,(\mathrm{N})}{2^k}$ are in] 0 $2^{\mathrm{N}-k}$ [

And as $Cy\,(\mathrm{N}) < \frac{2^{0.95\,\mathrm{N}}}{\mathrm{N}} = 2^{0.95\,\mathrm{N} - \mathrm{Ln}(\mathrm{N})/\mathrm{Ln}2}$ for N "approximation" of X

then $\frac{Cy\,(\mathrm{N})}{2^k} < 1 \Leftrightarrow Cy\,(\mathrm{N}) < 2^k \Leftrightarrow k > 0.95\,\mathrm{N} - \frac{\mathrm{Ln}(\mathrm{N})}{\mathrm{Ln}2}$

All v_0 should be greater than à $2^{\mathrm{N}-0.95\,\mathrm{N}} = 2^{0.05\,\mathrm{N}}$, or perhaps there are marginally a few exceptions in the very last values of k, when $k \approx 0.95\,\mathrm{N}$ because the number of remaining values is too small to have a perfect result... the $\frac{\mathrm{Ln}(\mathrm{N})}{\mathrm{Ln}2}$ steps can even still be used as a margin of error.

Let's make a more rigorous argument to upper bound the number of remaining transition lists (for which $v_0 < 2^{\mathrm{N}-k}$) :

We can see $d_{\mathrm{N}-1} = 1$ as a Bernouilli trial of parameter $p = \frac{1}{2}$, optimal parameter for the application of statistical results.

For the last transition :

If we consider X_n, the number of times that $d_{\mathrm{N}-1} = 1$ (for transition $\mathrm{N}-1$) for all the $n = Cy\,(\mathrm{N})$ lists of transitions, it is then a binomial distribution and $Z_n = \frac{X_n - \frac{n}{2}}{\sqrt{p\,(1-p)\,n}} = \frac{X_n - \frac{n}{2}}{\sqrt{\frac{1}{2}(1-\frac{1}{2})\,n}} = \frac{X_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$

converges to the standard normal distribution (for $n > 30$ because $p = \frac{1}{2}$).

Moreover, we have $\mathrm{P}(Z_n \le 4) > 0.9999$ (since in the tables of the standard normal distribution, the value is 1.0000)

We can consider that we are in this case to upper bound the number of remaining v_0, i.e. the number of $v_0 < 2^{\mathrm{N}-1}$ for the candidate transition lists. If that were not the case, it is not a problem since we will apply the same reasoning for the previous transitions.

However $Z_n \le 4 \Leftrightarrow X_n < \frac{n}{2} + 4\sqrt{\frac{n}{4}} = \frac{n}{2} + 2\sqrt{n} = \frac{n}{2}(1 + 4n^{-1/2})$
Let A_1 this upper bound of X_n and $A_0 = n$ the initial number of candidate lists.

If $n > 2^{10}$ then we can always use the convergence towards the standard normal distribution (because the limit of use is close to $n > 2^5$) and $u = 4n^{-1/2} < 2^{-3}$
We can always use inequality $1 + u < \mathrm{e}^u$ even if the upper bound becomes imprecise for $u = 0.125$ because $\frac{\mathrm{e}^{0.125}}{1.125} \approx 2.74$

$A_1 = \frac{n}{2}(1+u) < \frac{n}{2}\mathrm{e}^u = \frac{n}{2}\mathrm{e}^{4n^{-1/2}} = R_1$, the upper bound using the exponential which allows an easier chaining
We can follow exactly the same reasoning with the transition $\mathrm{N}-2$ with the same formalism, as long as the number of remaining lists is sufficient.
We have $R_2 \le \frac{R_1}{2}\mathrm{e}^{4R_1^{-1/2}}$

However $R_1^{-1/2} = (\frac{n}{2} e^{4n^{-1/2}})^{-1/2} = (\frac{n}{2})^{-1/2} e^{-2n^{-1/2}} < (\frac{n}{2})^{-1/2}$ because $-2n^{-1/2} < 0$

Therefore $R_2 \le \frac{n}{4} e^{4n^{-1/2}} e^{4(n/2)^{-1/2}} = \frac{n}{4} e^{4n^{-1/2}} e^{4n^{-1/2}\sqrt{2}} = \frac{n}{4} e^{4n^{-1/2}(1+\sqrt{2})}$

Starting again with the same formalism for d_{N-k}, as long as we can use convergence towards the standard normal distribution, we have :

$$R_k \le \frac{n}{2^k} e^{4n^{-1/2} S_k} \text{ with } S_k = \sum_{i=1}^{k} \sqrt{2}^{(i-1)} = \frac{\sqrt{2}^k - 1}{\sqrt{2}-1} < \frac{\sqrt{2}^k}{\sqrt{2}-1} = \frac{\sqrt{2^k}}{\sqrt{2}-1}$$

This result can be proven by induction, the formula being verified for $k = 1$ and $k = 2$.

If we assume that it is true for k, we prove that it is true for $k+1$

$R_{k+1} \le \frac{R_k}{2} e^{4R_k^{-1/2}}$

However $R_k^{-1/2} = \left(\frac{n}{2^k} e^{4n^{-1/2} S_k}\right)^{-1/2} = \left(\frac{n}{2^k}\right)^{-1/2} e^{-2n^{-1/2} S_k} < \left(\frac{n}{2^k}\right)^{-1/2} = n^{-1/2}\sqrt{2}^k$ because $-2n^{-1/2} S_k < 0$

Therefore $R_{k+1} \le \frac{n}{2^{k+1}} e^{4n^{-1/2} S_k} e^{4n^{-1/2}\sqrt{2}^k} = \frac{n}{2^{k+1}} e^{4n^{-1/2} S_{k+1}}$, the property is verified

We can improve the expression of R_k with the last upper bound of S_k

$R_k \le \frac{n}{2^k} e^{4\sqrt{2^k}/((\sqrt{2}-1)\sqrt{n})}$

Let K such as $2^{K+9} \le n \le 2^{K+10}$, we lower bound n by a power of 2.

We are assured that $R_{K-1} > 2^{10}$ therefore we can use the formula related to the convergence towards the standard normal distribution.

$$R_K \le \frac{n}{2^K} e^{4\sqrt{2^K}/((\sqrt{2}-1)\sqrt{2^{K+10}})} = \frac{n}{2^K} e^{1/(2^3(\sqrt{2}-1))} \approx 1.352 \times \frac{n}{2^K} < 2 \times \frac{n}{2^K} < \frac{2 \times 2^{K+10}}{2^K} = 2^{11} = 2048$$

i.e., when we have taken into account the K last transitions, the maximum relative error is 35.2% compared to the approximate value $\frac{n}{2^K}$, which makes the error less than a "step" !

We can say, by analyzing the K last transitions, that the number of values of $v_0 < 2^{N-K}$ is less than 2048.

The upper bound $1+u < e^u$ is still true but becomes too large.

However, until $R_k > 30$, the convergence towards the standard normal distribution is theoretically valid and this allows us to use the decreasing upper bound for a few more steps without using the exponential function.

We then have $R_K < 2048$ (we take $R_K = 2048$) and for $k > K$, we use the upper bound obtained at the beginning of the reasoning $R_{k+1} = \frac{R_k}{2}(1+4R_k^{-1/2})$ as long as $R_k > 30$

We then obtain the following decrease :

$k-K$	R_k
0	2048
1	1115
2	625
3	363
4	220
5	140

6	94
7	67
8	50
9	40
10	33
11	28

We can say, by analyzing the $K+11$ last transitions, that the number of values of $v_0 < 2^{N-(K+11)}$ is less than 30.

Then, we have no mathematical basis for conclude, apart from the fact that the Syracuse sequence is not a random process but a deterministic process. Indeed, once the lists of transitions have been determined, testing them any number of times will always give the same results. And the mechanism has no reason to change because it remains a reduced set.

Let's examine, the maximum possible number of transitions k so that no more value of v_0 is less than $2^{0.04\,N}$ or that none of the remaining 30 values remain after $0.96\,N$ steps.

We have $2^{K+9} \leq n = Cy\,(N) < 2^{0.95\,N} \leq 2^{K+10}$, donc $K+9 < 0.95\,N$ then $K+11 < 0.95\,N+2$
We are looking for the maximum value of k such that $K+11+k < 0.95\,N+2+k \leq 0.96\,N$

then $k < 0.01\,N-2$

In the case of the standard Syracuse sequence, the minimum length of a potential cycle for v is greater than or equal to 114208327604, then $k > 10^9$ whereas 5 would be sufficient in absolute terms because $30 < 32 = 2^5$, even if 2 or 3 additional steps, at most, could be possible.

For $b_1 \neq 1$, as $N > 1000$ (1539 for $v_0 > 0$ and 1054 for $v_0 < 0$), we have at least 13 or 8 steps respectively. Moreover, if N corresponds to an "approximation" of X, we still have $\frac{\mathrm{Ln}\,(N)}{\mathrm{Ln}2} = K+9 > 9$ possible additional steps !

We can therefore affirm that the minimum value of v_0, for all candidate transition lists, is greater than $2^{0.04\,N}$
We will see experimentally that the modeling by taking the remaining number equal to $\frac{1}{2^k}$ is very robust with a linear correlation coefficient more than 0.999 in absolute value if we consider the natural logarithm of the remaining count.

For $N = 1054$ and $v_0 < 0$, we have $Cy\,(N) < 2^{986}$, which means that the minimum value of v_0 is greater than 2^{50}

That's already sufficient, but let's reconsider the reasoning differently by eliminating the computational bias that involves reducing the number of transition lists to be tested and therefore to consider minimum v_0 (v_0 maximum for $v_0 < 0$)
Indeed, if we consider the C_N^m transition lists, then to have a cycle of length N, it is then necessary that N values of v_0 (the N elements of the cycle, since v_0 is no longer minimum (v_0 maximum for $v_0 < 0$)) be solutions.

As $N > 1000$ with the tests performed for the verification of the sequence, we can easily use the upper bound with the exponential for the K last steps and the convergence towards the standard normal distribution for the following 2 transitions which make it possible to guarantee that the number of values of v_0 is approximately $625 < 1000 < N$ from the previous table.

Therefore, the result is ensured by taking into account the $K+2$ latest transitions.

As we have $2^{K+9} < C_N^m < 2^{0.95\,N} \le 2^{K+10}$, then $K+2 < 0.95\,N - 7 < 0.96\,N$ for $N > 1000$ (with a few units of margin which are almostequivalent to dividing the remaining count by 2 each time) then all the elements of a cycle are greater than $2^{0.04\,N}$

All values of v_0, from which we could have a cycle, satisfy $v_0 \ge 2^{0.04\,N}$ for N > 1000 and for $|b_1| < 1024$.

Remark : This reasoning can be generalized by considering $\frac{v_N}{v_0} \approx \frac{3^m}{2^N}$ (not necessarily close to 1). It is then sufficient to consider the C_N^m lists of transitions $L(N, m, d)$ and we can have a distribution of the v_0 solution as well as an order of magnitude of the minimum value.

5. **Summary on the existence of cycles**

According to the paragraph VI-1-2, summary of the minimum length of a cycle, we saw that the maximum value of v_0 was less than $vMax\ = 2^{f(N)+B_1}$ with $f(N) = \frac{2}{\text{Ln}2}\text{Ln}(N)+4$ (even if we know that $vMax$ increases by jumps at each "approximation" of X and is much lower for the other values of N)

According to the conclusion of the previous paragraph VI-4-4, we have that the minimum value of v_0, for all the transition lists that may correspond to a cycle, was greater than $vMin\ = 2^{0.04\,N}$ in the case where N > 1000 for $|b_1| < 1024$.

If $vMax\ < vMin$, then there can be no cycle.

Which is equivalent to $g(N) = 0.04\,N - \frac{2}{\text{Ln}2}\text{Ln}(N) - 4 > B_1$

We derive to study variations on $[\ 1000 \quad +\infty\ [$: $g'(N) = 0.04 - \frac{2}{\text{Ln}2\,N} > 0$ for $N > \frac{2}{0.04\,\text{Ln}2} = 73$ (then g is increasing on $[\ 73 \quad +\infty\ [$)

However $g(1000) \approx 16.06$ which is therefore the minimum of g on $[\ 1000 \quad +\infty\ [$

In the standard case, as $N \ge 114208327604$, we have $g(114208327604\ ) = 4568333026$, then there is no other cycle.

As $B_1 \le 10$ for $|b_1| < 1024$, inequality is verified. As $g(1539) = 36.38$ and $g(1054) = 18.07$, this would still leave at least 26 or 8 additional steps, for cases N = 1539, respectively N = 1054, for the approximation of $vMin$ and therefore there are no other cycles than the trivial cycles.

However, the upper bounds are large because for N = 1054 and $v_0 < 0$, we have $Cy\,(N) < 2^{986}$, which means $> 2^{50}$ or $vMax\ < 2^{22+B1} < 2^{32}$ in all cases of $|b_1| < 1024$
We could also have tested the sequences with $N_0 = 22$ instead of $N_0 = 17$ and minimum lengths would have been N = 14187 for $v_0 > 0$ and N = 25781 for $v_0 < 0$, which would have required to a bit more preliminary calculations but would also have allowed for a much more comfortable margin (more than 700 because $C(14187) = 2^{13456.06}$ and $C(25781) = 2^{24468.55}$ while $vMax\ < 2^{28+B1}$).

Conclusion : There can not be any cycle other than the trivial cycles for $|b_1| < 1024$.

We can notice that we do not use the necessary and sufficient condition, which is even more restrictive, namely $v_N = v_0$
It's already quite extraordinary that by fixing only 5% of the bits (0.05×N low-order bits set to 0 or 1), we can

have $\frac{v_N}{v_0} \approx 1$, ie the last 95% transitions are done naturally.

There exists no other cycle, but values of v_0 can be found that create "almost" cycles with relative error as low as we want; the details are in the paragraph IX-1, this is "Theorem 1".

6. **Experimental results**

We randomly test candidate transition lists $L(N, m, d)$ to possibly obtain a cycle even if for some values of N, there could not be a cycle except for $b_1 \neq 1$.
The chosen transition lists are "above" JGL(N) and we can see the distribution of v_0, this is what primarily interests us.

For small values of N, the calculations are exhaustive, which makes it possible to see the evolution of r_{N_k} at the end, in real condition, to see that it is a very deterministic process, far from a coin toss.

Obviously, for larger values of N, the number of lists tested is very low, the test is not exhaustive, far from it !... but this allows us to see that on random samples, the statistical method for the distribution of v_0 is very robust (linear correlation coefficient greater than 0.999)

In the results concerning the distribution of v_0, here is what represents :

- nb : the number of lists to test for N, i.e. 312455 for N = 27
- e_k, the effective number of v_0 such that $2^{N-k} \leq v_0 < 2^{N-k+1}$ (i.e. N - NbBits(v_0) = $k-1$)
- re_k, the effective number of v_0 such that $v_0 < 2^{N-k}$
- $A_k = \frac{re_{k-1}}{2} + 2\sqrt{re_{k-1}}$, the approximation of re_k at 4 standard deviations by approximating with a standard normal distribution
- $R_k = \frac{nb}{2^k} e^{4(\sqrt{2}^k - 1)/(\sqrt{2} - 1)/\sqrt{nb}}$ with 4 standard deviations by accumulating the approximations
- $q_k = \frac{R_k}{\frac{nb}{2^k}}$, the ratio with the theoretical central value

Below the following section allowing the tests (same code as for the sequence test at the end of the document), are the complete results for N = 27

Choose the value of b_1 : [ ] OK (odd relative integer, b_1 can be negative and v_0 is always positive)

Random tests for cycles : N : 17 pendant 1 minute Try Stop Previous results
☑ Statistics on v_0
☐ Addition of statistics on the transitions
☐ Addition of statistics on r_N (longer calculations)

Found cycles (v_0 = 0 is eliminated and if b1 < 0 then v_0 = -b1 is also omitted) :

Complete results for N = 27 and $b_1 = 1$
List of elements of results :

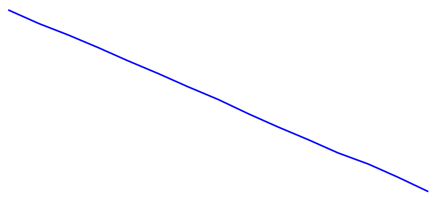
1. N
2. Number of lists of transitions computed (for cycle)
3. Time for computations
4. Minimum value of v(0) and description : (list of transitions, v(0), v(N))
5. Number of times, we get n for "v = N - NbBits(v(0))" (starting from 0)
6. Average and deviation for v
7. Statistics on transitions
8. Number of times, we get n for "r = NbBits(r_N)" (starting from 0)

9. Average and deviation for r

---

1. 27
2. 312455
3. 8.385 s
4. 251 : ("110110111111001111000101010",251,244)
5. 156040,78398,38861,19543,9806,4872,2481,1269,591,298,151,67,41,20,10,4,0,1,1,1

   The number of tests = 312455 < 2^{19} and max(v) = 20. The distribution is not exactly as expected, but the difference is low. N = 27 is quite small too.
   Graphics of $Ln(re_k)$ for k less than first time $r_k \le 10$:

   Coefficient of correlation R ~ -0.9999622845169954

k	e_k	re_k	A_k	$re_k < A_k$	R_k	q_k
0	0	312455				
1	156040	156415	157345	true	157349	1.007
2	78398	78017	78998	true	79474	1.017
3	38861	39156	39567	true	40310	1.032
4	19543	19613	19973	true	20567	1.053
5	9806	9807	10086	true	10582	1.084
6	4872	4935	5101	true	5509	1.128
7	2481	2454	2607	true	2917	1.195
8	1269	1185	1326	true	1581	1.295
9	591	594	661	true	---	---
10	298	296	345	true	---	---
11	151	145	182	true	---	---
12	67	78	96	true	---	---
13	41	37	56	true	---	---
14	20	17	30	true	---	---
15	10	7	16	true	---	---
16	4	3	8	true	---	---
17	0	3	4	true	---	---
18	1	2	4	true	---	---
19	1	1	3	true	---	---
20	1	0	2	true	---	---

6. 1.00099 and 1.41447
7.

n	t_n=0 && d_n=0	t_n=0 && d_n=1	t_n=0	% d_n=1	t_n=1 && d_n=0	t_n=1 && d_n=1	t_n=1	% d_n=1
0	0	0	0	0	0	312455	312455	100
1	0	0	0	0	0	312455	312455	100
2	58040	0	58040	0	0	254415	254415	100
3	58040	0	58040	0	0	254415	254415	100
4	58040	0	58040	0	58040	196375	254415	77.18
5	69473	11433	80906	14.13	46607	184942	231549	79.87
6	58040	22866	80906	28.26	70348	161201	231549	69.61
7	58040	22866	80906	28.26	81781	149768	231549	64.68
8	49259	50211	99470	50.47	123909	89076	212985	41.82

9	49259	50211	99470	50.47	118605	94380	212985	44.31
10	59588	46182	105770	43.66	99718	106967	206685	51.75
11	56609	49161	105770	46.47	97887	108798	206685	52.63
12	51065	54705	105770	51.72	107061	99624	206685	48.2
13	61721	53654	115375	46.5	94723	102357	197080	51.93
14	61582	53793	115375	46.62	94794	102286	197080	51.9
15	56601	58774	115375	50.94	99839	97241	197080	49.34
16	63418	66237	129655	51.08	92788	90012	182800	49.24
17	64240	65415	129655	50.45	91354	91446	182800	50.02
18	68102	68280	136382	50.06	87698	88375	176073	50.19
19	68802	67580	136382	49.55	86831	89242	176073	50.68
20	68439	67943	136382	49.81	87862	88211	176073	50.09
21	76887	75585	152472	49.57	79492	80491	159983	50.31
22	76832	75640	152472	49.6	79685	80298	159983	50.19
23	75877	76595	152472	50.23	80471	79512	159983	49.7
24	101336	102169	203505	50.2	55005	53945	108950	49.51
25	101675	101830	203505	50.03	54313	54637	108950	50.14
26	156415	156040	312455	49.93	0	0	0	0

8. 281,154199,157974,1
9. 1.5047 and 0.50177

The case N = 27 is representative of what is happening in the last steps of *k* because basically, we already only have 312455 lists to test, which is little.

The results obtained for N = 27 are really very close to statistical theoretical modeling, without even using the upper bound of the remaining count with 2 standard deviations.

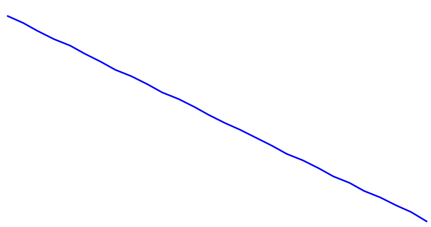
We see a linear correlation coefficient between *k* and $\mathrm{Ln}(re_k)$ close to -0.99996, which is excellent (even for values of re_k up to 10, which is lower than the theoretical convergence of the model which is limited to about 30)

For the last values of *k*, the decrease is very mechanical, there is no real random phenomenon.

Partial results for N = 40 and $b_1 = 1$
We focus here on the distribution of v_0 (the result indexed by 5) :

We get :

5. 949744268,474866457,237419589,118721656,59361118,29681430,14840202,7419771,3709525,1855537,927725
   The number of tests = 1899474678 < 2^{31} and max(v) = 34. The distribution is not exactly as expected, but the difference is low. N = 40 is quite small too.
   Graphics of $\mathrm{Ln}(re_k)$ for k less than first time $r_k \leq 10$:



Coefficient of correlation R ~ -0.9999822863394272

k	e_k	re_k	A_k	$re_k < A_k$	R_k	q_k
0	0	1899474678				
1	949744268	949730410	949824504	true	949824508	1
2	474866457	474863953	474926840	true	474973899	1

3	237419589	237444364	237475559	true	237530546	1
4	118721656	118722708	118753000	true	118796107	1.001
5	59361118	59361590	59383145	true	59419863	1.001
6	29681430	29680160	29696204	true	29725360	1.002
7	14840202	14839958	14850975	true	14873597	1.002
8	7419771	7420187	7427683	true	7444524	1.003
9	3709525	3710662	3715541	true	3727732	1.005
10	1855537	1855125	1859183	true	1867740	1.007
11	927725	927400	930286	true	936617	1.01
12	463941	463459	465626	true	470257	1.014
13	231999	231460	233091	true	236514	1.02
14	115918	115542	116692	true	119243	1.029
15	57567	57975	58450	true	60326	1.041
16	28996	28979	29469	true	30668	1.058
17	14479	14500	14829	true	15698	1.083
18	7377	7123	7490	true	8114	1.12
19	3566	3557	3730	true	4252	1.174
20	1744	1813	1897	true	2272	1.254
21	930	883	991	true	---	---
22	436	447	500	true	---	---
23	211	236	265	true	---	---
24	110	126	148	true	---	---
25	63	63	85	true	---	---
26	31	32	47	true	---	---
27	18	14	27	true	---	---
28	8	6	14	true	---	---
29	4	2	7	true	---	---
30	1	1	3	true	---	---
31	0	1	2	true	---	---
32	0	1	2	true	---	---
33	0	1	2	true	---	---
34	1	0	2	true	---	---

Looking at these results for N = 40, we see the numbers of occurrences for v_0 which are at the end "31,18,8,4,1,0,0,1" and so there is 1 value of v_0 which falls outside the interval with 3 bits less than what is theoretically expected. Thus, the gap is insignificant.

VII. **Study of the divergence towards infinity**

The method used to have the minimum value of v_0 thanks to the statistical distribution of the v_0 is very robust.

We will proceed in the same way to study a possible divergence

- We will count or at least upper bound the number of lists for which $v_n > v_0$ for all n ($v_n < v_0$ for $v_0 < 0$). As a reminder, proving the theorem is equivalent to identifying the cycles and proving that there is no divergence towards infinity for v, then for u
- We will eliminate lists for which there are potential cycles from v_0 less than the minimum value of a trivial cycle
- We will conclude with the statistical distribution to prove that there can be no divergence
- We will verify the consistency with the records of "altitude flight" (glide)
- We will detail the results of my completely useless test of the standard sequence for $v_0 < 2^{40}$ pour $b_1 = 1$
- We will add the possibility of testing the sequence

1. **The number *Up* (N) of transition lists of length N such as, for $v_n > v_0$ for $n \le$ N, is less than or equal to $C_N^{m_N}$**

We are interested in transition lists L of length N such that, for all $0 < n \le N$, $v_n > v_0$ ($v_n < v_0$ for $n \le N$ if $v_0 < 0$) because :

- Thanks to the "Theorem 0" (paragraph VI-2), they each characterize a minimum solution $v_0 < 2^N$ (bijection) and it is only these values of v_0 that must be tested to verify the sequence by calculation up to 2^N. We are sure to do the minimum of calculations.
- We then also have the distribution of the v_0, new element for the reasoning. Indeed, the probability that $2^{N-(k+1)} \le v_0 < 2^{N-k}$ is $\frac{1}{2^k}$ and their number therefore tends towards $\frac{Up\ (N)}{2^k}$.
- These lists are easily obtained thanks to the boundary transition list JGL(n), already defined above in the document (sections IV and V).
  At each transition t_n (which is indexed from 0), it is simply necessary that the number of "type 1" transitions of L is greater than or equal to that of JGL($n+1$) (which is indexed from 1).
  Compared to the previous section where it was a matter of finding cycles, this time, the value of N does not necessarily correspond to an "approximation" of X and the number of "type 1" transitions is not limited and can therefore be equal to N.
- We can easily calculate the exact number of *Up* (N) but an upper bound (even wide) allows a statistical reasoning.

Method 1 : A recursion is used as for cycles, with no upper limit for m, this would also make it possible to generate the transition lists and to perform the processing. The downside is that it's slow.
`Javascript code not available in this document.`
Method 2 : We use the same method as for cycles, method with memorization of the number of lists $\text{Nb}(n, m)$ for a length n and for a given number m of "type 1" transitions (without upper limit for m).

We have :

- $\text{Nb}(0, 0) = 1$
- $\text{Nb}(0, k) = 0$ for $1 \le k \le N$
- The elements are constructed for $n+1$ from those of n as follows :
- $\text{Nb}(n+1, k) = 0$ for $0 \le k \le N$
- $\text{Nb}(n+1, k)$ += $\text{Nb}(n, k-1)$ for $0 < k \le N$ (adding a "type 1" transition is always possible)
- $\text{Nb}(n+1, k)$ += $\text{Nb}(n, k)$ for $k \ge m_{n+1}$ (possible addition of a transition of "type 0")
- The number of lists that interests us is $Up\ (N) = \sum_{n=m_N}^{N} \text{Nb}(N, n)$

We do not generate the lists but it is much faster.

We have the exact number but it's still a result by induction, we don't have a direct formula.

`Javascript code not available in this document.`

Length N : 17 Try

We get the following results :

N	*Up* (N)
10	$\sim 2^{6} \sim 2^{0.6 \times N} \sim 2^{N-4} = 64$
17	$\sim 2^{12.0458} \sim 2^{0.7086 \times N} \sim 2^{N-4.9542} = 4228$
20	$\sim 2^{14.7381} \sim 2^{0.7369 \times N} \sim 2^{N-5.2619} = 27328$
30	$\sim 2^{23.6064} \sim 2^{0.7869 \times N} \sim 2^{N-6.3936} = 12771274$
40	$\sim 2^{32.5761} \sim 2^{0.8144 \times N} \sim 2^{N-7.4239} = 6402835000$

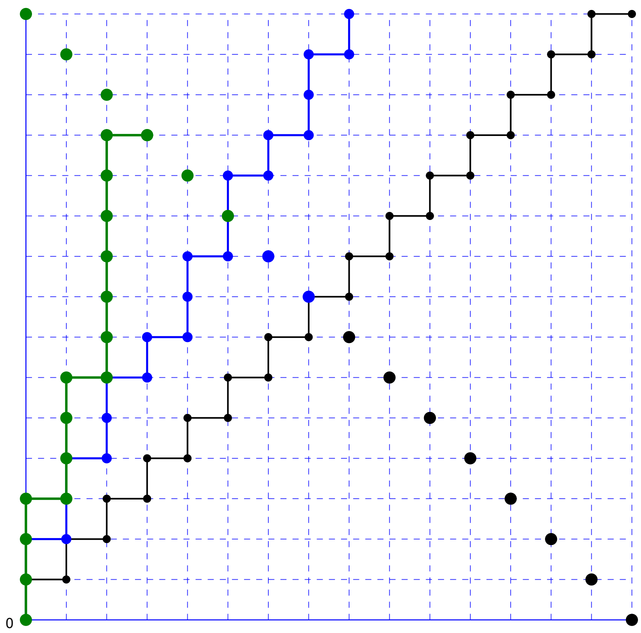
50	$\sim 2^{41.764} \sim 2^{0.8353 \times N} \sim 2^{N-8.236} = 3734259929440$
60	$\sim 2^{50.977} \sim 2^{0.8496 \times N} \sim 2^{N-9.023} = 2216134944775156$
68	$\sim 2^{58.3377} \sim 2^{0.8579 \times N} \sim 2^{N-9.6623} = 364249198012174112$
69	$\sim 2^{59.2538} \sim 2^{0.8588 \times N} \sim 2^{N-9.7462} = 687342103066248112$
70	$\sim 2^{60.1068} \sim 2^{0.8587 \times N} \sim 2^{N-9.8932} = 1241503538986719152$
80	$\sim 2^{69.4443} \sim 2^{0.8681 \times N} \sim 2^{N-10.5557} = 803209913882910595105$
90	$\sim 2^{78.7507} \sim 2^{0.875 \times N} \sim 2^{N-11.2493} = 508520069189622659715764$
100	$\sim 2^{87.9674} \sim 2^{0.8797 \times N} \sim 2^{N-12.0326} = 302560669500543257546172187$

Method 3 : We upper bound the number of lists thanks to Catalan's triangle[5][6]

As for the cycles, we can create a diagram by representing the number of "type 1" transitions on the vertical axis. Transition lists are then paths with movements upward or to the right (toward the North or toward the East). The valid transition lists are those that remain above the JGL boundary and end after N steps at the diagonal points (on the line of equation $y = N - X$) above the one corresponding to JGL(N).

In the diagram below :

- The blue line represents the boundary JGL
- The green line corresponds to a valid transition list because it is always above JGL
- The black line corresponds to the boundary for the Catalan's triangle
- Green dots are the end points of valid transition lists
- The blue dots are the end points of the transition lists that are not valid but which are for the Catalan's triangle
- The black dots are the end points of the transition lists that are not valid even for the Catalan's triangle



As the boundary JGL is always "above" the diagonal, the constraint is more restrictive (perhaps by reversing the axes relative to the conventions)

Nonetheless, $Up\,(N) < \sum_{i=0}^{d_N} C(m_N + i, d_N - i)$ with $C(n, p)$, numbers of the Catalan's triangle

We have the formula for $p > 0$: $C(n, p) = \binom{n+p}{n} - \binom{n+p}{n+1}$ and $C(n, 0) = 1$

Or for $p > 0$: $C(n, p) = C_{n+p}^{n} - C_{n+p}^{n+1}$ with the old combination notation and $C(n, 0) = 1$

Then $Up\,(\mathrm{N}) < \sum_{i=0}^{d_\mathrm{N}} \mathrm{C}(m_\mathrm{N}+i, d_\mathrm{N}-i) = \mathrm{C}(\mathrm{N},0) + \sum_{i=0}^{d_\mathrm{N}-1} \mathrm{C}(m_\mathrm{N}+i, d_\mathrm{N}-i)$

$Up\,(\mathrm{N}) < 1 + \sum_{i=0}^{d_\mathrm{N}-1} (\mathrm{C}_\mathrm{N}^{m_\mathrm{N}+i} - \mathrm{C}_\mathrm{N}^{m_\mathrm{N}+i+1})$

We have a telescoping of terms and

$Up\,(\mathrm{N}) < 1 + \mathrm{C}_\mathrm{N}^{m_\mathrm{N}} - \mathrm{C}_\mathrm{N}^{\mathrm{N}} = \mathrm{C}_\mathrm{N}^{m_\mathrm{N}}$

According to the study made in the paragraph VI-4-1-3, $Up\,(\mathrm{N}) < 2^{0.95\,\mathrm{N}}$, for $\mathrm{N} > 1000$, which is a very loose upper bound (in the absence of an equivalence).

$Up\,(\mathrm{N}) < \mathrm{C}_\mathrm{N}^{m_\mathrm{N}} < 2^{0.95\,\mathrm{N}}$ for $\mathrm{N} > 1000$.

2. **Candidate transition lists to eliminate**

The transition lists for which there are potential cycles from values of v_0 less than the minimum value of a trivial cycle, are part of the lists counted in the previous paragraph.

However, their solutions v_0 (minimum or not) will inevitably lead to a cycle and do not represent a possibility of divergence towards infinity. So you have to eliminate them.

In the case of the standard Syracuse sequence, there is only one cycle for v (of length 2 consisting of values 1 and 2) and the "10..." or "01..." lists that would lead to the cycle for v are eliminated because the JGL boundary list begins with "11", which imposes $v_0 \geq 3$ for all candidate lists of transitions (for which $v_n \geq v_0$ for all n). So there is no problem.

For the same reason, the trivial cycle of length 2 for v from $v_0 = b_1$, for $v_0 > 0$ is eliminated with JGL structure and constraint on the list.

On the other hand, in the case $v_0 < 0$, the trivial cycle of length 1 with $v_0 = -b_1$ is not eliminated because the transition list is only 1 i.e. "1..."

Let's now study these different possibilities for $|b_1| < 1024$, based on the results of tests of the sequence for $v_0 < 2^{N_0+B_1}$

We limit ourselves to cases where v_0 and b_1 are coprime, $\gcd(v_0, b_1) = 1$

It is enough to test, for each trivial cycle, if, from a value v lower than the minimum value of the cycle v_0, we can reach this cycle because in such a case, we have $v_n > v$ for all n.

The table lists all these cases.

Table structure :

b_1	v_0 (minimum value of the cycle)	v (value from which this cycle can be reached)

For $b_1 > 0$ See/Hide results

Results at the end of the document

See the results

For $b_1 < 0$ See/Hide results

Results at the end of the document

See the results

We can note that the cases are very rare and only represent a few lists at most (nevertheless 71 lists for $b_1 = -311$) and only for some values of b_1.

We can also remove all the possibly remaining lists that have as their solution the minimum value of a trivial cycle... which adds only a few lists

3. **Conclusion about the existence of divergences towards infinity**

In paragraph VI-4-4, we obtained an upper bound of the number of $v_0 < 2^{N-k}$ which was very close to $\frac{1}{2^k}$ for the study of the cycles.

Using exactly the same statistical reasoning, this time with all the transition lists of length N which allow to have $v_n \geq v_0$ for all $n \leq N$, we have :

If $2^{K+10} \leq Up\ (N) < 2^{K+11}$, then by taking into account the $K+12$ last steps, we are sure that the number of remaining lists is less than 30.

Moreover, as we have the same upper bound for Cy (N) and Up (N) by C_N^m, it can be said that :

All values of v_0, from which we could have $v_n \geq v_0$ for all $n \leq N$, verify $v_0 \geq 2^{0.04\,N}$ for N > 1000 and for $|b_1| < 1024$.

Let's make a simple proof by contradiction :

Assume that there exists a value v_0 for which the v sequence diverges.

Then, there exists a value p for which $v_n \geq v_p$ for all $n \geq p$ with $2^{N-1} \leq v_p < 2^N$
This allows us to consider the case of the transition lists studied previously for a length greater than p of which v_p should always be the solution of one of them, regardless of the envisaged length (this eliminates the elements for $0 < n < p$ such as $v_n < v_0$).

It is enough to take $N_1 = 10000\,N$ (for example, there is no need to search the minimum value of k with $N_1 = k\,N$).

We then have the minimum value of all these lists of transitions which should be greater than $2^{0.04 N_1} = 2^{400\,N} > v_p$ (with a very large margin)

Thus, we have a contradiction; it is therefore impossible

There is no divergence to infinity for $|b_1| < 1024$.

4. **Consistency with the records of "altitude flight" (glide)**

We call N, length of a "altitude flight" (glide), the number of terms for which $v_n > v_0$.
Then N is such that for all $0 < n < N$, $v_n > v_0$ and $v_N < v_0$
We call a "record" for v_0 if for any value less than v_0, the "altitude flight" (glide) is lower than that of v_0
In Eric Roosendaal's document[7], we can find the list of records.

We take back this list and in this table, for each record, we put the number v_0 in the form 2^n and we calculate the flight time N for v (instead of u) and if $0.04\,N < n$ then v_0 is well above the minimum bound $2^{0.04\,N}$ (and therefore it is consistent)

We also calculate $Up\ (N) \sim 2^{N-p}$ and we see that n is close to p, then there is $p-n$ step difference for the record compared to a simple division by 2 of the number of lists at each step, which confirms the modeling and the very mechanical and absolutely non-random nature.

Index	N_u for u	$v_0 = 2^n$	n	N for v	$0.04\,N$	$0.04\,N < n$	Up (N)	p	$p-n$
34	1639	2602714556700227743	61.1747	1005	40.2	true	$\sim 2^{943.0839} \sim 2^{0.9384 \times N} \sim 2^{N-}$	61.9161	0.7414

							61.9161		
33	1614	1236472189813512351	60.1009	990	39.6	true	$\sim 2^{928.9491} \sim 2^{0.9383\times N} \sim 2^{N-61.0509}$	61.0509	0.95
32	1575	180352746940718527	57.3236	966	38.64	true	$\sim 2^{906.1832} \sim 2^{0.9381\times N} \sim 2^{N-59.8168}$	59.8168	2.4932
31	1471	118303688851791519	56.7153	902	36.08	true	$\sim 2^{845.4672} \sim 2^{0.9373\times N} \sim 2^{N-56.5328}$	56.5328	-0.1825
30	1445	1008932249296231	49.8418	886	35.44	true	$\sim 2^{830.2837} \sim 2^{0.9371\times N} \sim 2^{N-55.7163}$	55.7163	5.8745
29	1187	739448869367967	49.3934	728	29.12	true	$\sim 2^{680.6457} \sim 2^{0.935\times N} \sim 2^{N-47.3543}$	47.3543	-2.0391
28	1177	70665924117439	46.0061	722	28.88	true	$\sim 2^{675.0009} \sim 2^{0.9349\times N} \sim 2^{N-46.9991}$	46.9991	0.993
27	1161	31835572457967	44.8557	712	28.48	true	$\sim 2^{665.4982} \sim 2^{0.9347\times N} \sim 2^{N-46.5018}$	46.5018	1.6461
26	1122	13179928405231	43.5834	688	27.52	true	$\sim 2^{642.7422} \sim 2^{0.9342\times N} \sim 2^{N-45.2578}$	45.2578	1.6744
25	988	2081751768559	40.9209	606	24.24	true	$\sim 2^{565.145} \sim 2^{0.9326\times N} \sim 2^{N-40.855}$	40.855	-0.0659
24	897	898696369947	39.709	550	22	true	$\sim 2^{512.0933} \sim 2^{0.9311\times N} \sim 2^{N-37.9067}$	37.9067	-1.8023
23	892	12235060455	33.5103	547	21.88	true	$\sim 2^{509.2836} \sim 2^{0.931\times N} \sim 2^{N-37.7164}$	37.7164	4.2061
22	729	2788008987	31.3766	447	17.88	true	$\sim 2^{414.6802} \sim 2^{0.9277\times N} \sim 2^{N-32.3198}$	32.3198	0.9432
21	706	1827397567	30.7671	433	17.32	true	$\sim 2^{401.4799} \sim 2^{0.9272\times N} \sim 2^{N-31.5201}$	31.5201	0.753
20	649	1200991791	30.1616	398	15.92	true	$\sim 2^{368.3885} \sim 2^{0.9256\times N} \sim 2^{N-29.6115}$	29.6115	-0.5501
19	644	217740015	27.698	395	15.8	true	$\sim 2^{365.5739} \sim 2^{0.9255\times N} \sim 2^{N-29.4261}$	29.4261	1.7281
18	613	63728127	25.9254	376	15.04	true	$\sim 2^{347.6308} \sim 2^{0.9246\times N} \sim 2^{N-28.3692}$	28.3692	2.4438
17	502	56924955	25.7626	308	12.32	true	$\sim 2^{283.4323} \sim 2^{0.9202\times N} \sim 2^{N-}$	24.5677	-1.1949

							24.5677		
16	486	26716671	24.6712	298	11.92	true	$\sim 2^{273.9535} \sim 2^{0.9193\times N} \sim 2^{N-24.0465}$	24.0465	-0.6247
15	476	20638335	24.2988	292	11.68	true	$\sim 2^{268.3451} \sim 2^{0.919\times N} \sim 2^{N-23.6549}$	23.6549	-0.6439
14	468	13421671	23.6781	287	11.48	true	$\sim 2^{263.5931} \sim 2^{0.9184\times N} \sim 2^{N-23.4069}$	23.4069	-0.2712
13	401	8088063	22.9474	246	9.84	true	$\sim 2^{224.9755} \sim 2^{0.9145\times N} \sim 2^{N-21.0245}$	21.0245	-1.9229
12	365	1126015	20.1028	224	8.96	true	$\sim 2^{204.2689} \sim 2^{0.9119\times N} \sim 2^{N-19.7311}$	19.7311	-0.3717
11	298	1027431	19.9706	183	7.32	true	$\sim 2^{165.7531} \sim 2^{0.9058\times N} \sim 2^{N-17.2469}$	17.2469	-2.7237
10	287	626331	19.2566	176	7.04	true	$\sim 2^{159.0993} \sim 2^{0.904\times N} \sim 2^{N-16.9007}$	16.9007	-2.3559
9	282	381727	18.5422	173	6.92	true	$\sim 2^{156.314} \sim 2^{0.9035\times N} \sim 2^{N-16.686}$	16.686	-1.8562
8	269	362343	18.467	165	6.6	true	$\sim 2^{148.7835} \sim 2^{0.9017\times N} \sim 2^{N-16.2165}$	16.2165	-2.2505
7	267	270271	18.044	164	6.56	true	$\sim 2^{147.9135} \sim 2^{0.9019\times N} \sim 2^{N-16.0865}$	16.0865	-1.9575
6	220	35655	15.1218	135	5.4	true	$\sim 2^{120.6796} \sim 2^{0.8939\times N} \sim 2^{N-14.3204}$	14.3204	-0.8014
5	171	10087	13.3002	105	4.2	true	$\sim 2^{92.6731} \sim 2^{0.8826\times N} \sim 2^{N-12.3269}$	12.3269	-0.9733
4	132	703	9.4574	81	3.24	true	$\sim 2^{70.2925} \sim 2^{0.8678\times N} \sim 2^{N-10.7075}$	10.7075	1.2501
3	96	27	4.7549	59	2.36	true	$\sim 2^{49.977} \sim 2^{0.8471\times N} \sim 2^{N-9.023}$	9.023	4.2681
2	11	7	2.8074	7	0.28	true	$\sim 2^{3.7004} \sim 2^{0.5286\times N} \sim 2^{N-3.2996}$	3.2996	0.4922
1	6	3	1.585	4	0.16	true	$\sim 2^{1.585} \sim 2^{0.3963\times N} \sim 2^{N-2.415}$	2.415	0.83

5. **My own verification of the sequence for $n \le 2^{40}$ with $b_1 = 1$**

Even if it has no interest, with my own computer resources, I was able to test Syracuse sequence up to $n \le 2^{40} = 1099511627776$

It is enough to test 6402835000 transition lists (0.58% of 2^{40}) in 49778 seconds or just a little less than 14 hours (without additional statistics) with an Intel(R) Core(TM) i5-1035G1 processor, Google Chrome "Version 119.0.6045.160 (Official Build) (64 bits)", with the JavaScript program included in this file, and we obtain the following final result.

The verification is performed on 1295 processes that could be launched in parallel on as many computers (their duration is not really balanced, ranging from 9 to 250 seconds, with an average of around 39 seconds).

Here are the test results after removing a visualization and a modeling of the behavior of lists having the same number of "type 1" transitions :
Global results for some parallel processes

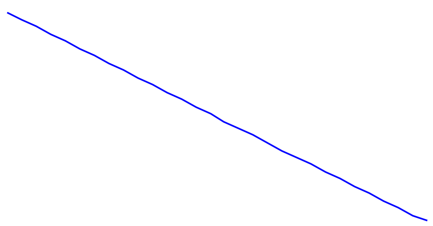
---

n1 : first value of n such as v(N+n) < v(0)
n2 : first value of n such as v(N+n) = 1

1. N
2. Number of lists of transitions computed (above JGL(40))
3. Time for computations
4. Minimum value of v(0) and description : (list of transitions, v(0), v(N) , n1)
5. Number of times, we get n for "v = N - NbBits(v(0))" (starting from 0)
6. Average and deviation for v
7. n1Max = Maximum number for n1
8. Description of a solution for n1Max : (list of transitions, v(0), v(N), n1)
9. Average and deviation for n1
10. Number of times, we get n for n1 (starting from 0)

---

1. 40
2. 6402835000
3. 49778.090999999986 s
4. 27 : ("1101111101011011101111010011101101111110",27,1822,19)
5. 3201415313,1600712335,800351093,400180065,200086844,100044150,50022007,25011669,12505491,625
   The number of tests = 6402835000 < 2^{33} and max(v) = 36. The distribution is not exactly as expected, but the difference is low. N = 40 is quite small too.
   Graphics of Ln(re_k) for k less than first time $r_k \le 10$:



Coefficient of correlation R ~ -0.999984102568291

k	e_k	re_k	A_k	$re_k < A_k$	R_k	q_k
0	0	6402835000				
1	3201415313	3201419687	3201577535	true	3201577539	1
2	1600712335	1600707352	1600823005	true	1600901941	1
3	800351093	800356259	800433693	true	800531002	1
4	400180065	400176194	400234710	true	400322098	1
5	200086844	200089350	200128105	true	200201076	1.001
6	100044150	100045200	100072965	true	100128848	1.001
7	50022007	50023193	50042604	true	50084449	1.001
8	25011669	25011524	25025741	true	25056391	1.002
9	12505491	12506033	12515764	true	12538220	1.003
10	6252597	6253436	6260089	true	6276205	1.004
11	3127616	3125820	3131719	true	3143126	1.005
12	1564356	1561464	1566445	true	1575122	1.008
13	781178	780286	783231	true	790084	1.011

14	390764	389522	391909	true	396833	1.015
15	195203	194319	196009	true	199690	1.022
16	97093	97226	98041	true	100752	1.031
17	48690	48536	49236	true	51025	1.045
18	24612	23924	24708	true	25978	1.064
19	12160	11764	12271	true	13326	1.091
20	5846	5918	6098	true	6908	1.131
21	2944	2974	3112	true	3635	1.191
22	1508	1466	1596	true	1954	1.28
23	723	743	809	true	---	---
24	358	385	426	true	---	---
25	187	198	231	true	---	---
26	101	97	127	true	---	---
27	46	51	68	true	---	---
28	27	24	39	true	---	---
29	10	14	21	true	---	---
30	6	8	14	true	---	---
31	1	7	9	true	---	---
32	0	7	8	true	---	---
33	0	7	8	true	---	---
34	3	4	8	true	---	---
35	2	2	6	true	---	---
36	2	0	3	true	---	---

6. 1 and 1.4142
7. 510
8. ("1101111111110101111111011011011110101010",898696369947,168287090689819,510)
9. 18.67363 and 19.45287

In another test, having checked "Add transition statistics", we obtain the additional results :

n	t_n=0 && d_n=0	t_n=0 && d_n=1	t_n=0	% d_n=0	t_n=1 && d_n=0	t_n=1 && d_n=1	t_n=1	% d_n=1
0	0	0	0	0	0	6402835000	6402835000	0
1	0	0	0	0	0	6402835000	6402835000	0
2	1100914210	0	1100914210	100	0	5301920790	5301920790	0
3	1100914210	0	1100914210	100	0	5301920790	5301920790	0
4	1100914210	0	1100914210	100	1100914210	4201006580	5301920790	20.76
5	1298043035	197128825	1495171860	86.81	903785385	4003877755	4907663140	18.41
6	1100914210	394257650	1495171860	73.63	1413313120	3494350020	4907663140	28.79
7	1100914210	394257650	1495171860	73.63	1610441945	3297221195	4907663140	32.81
8	944159584	833631669	1777791253	53.1	2469110961	2155932786	4625043747	53.38
9	944159584	833631669	1777791253	53.1	2388362563	2236681184	4625043747	51.63
10	1116734963	745348718	1862083681	59.97	2002749808	2538001511	4540751319	44.1
11	1042500750	819582931	1862083681	55.98	2067140986	2473610333	4540751319	45.52
12	909215511	952868170	1862083681	48.82	2337924822	2202826497	4540751319	51.48
13	1105114649	865318617	1970433266	56.08	2024375455	2408026279	4432401734	45.67
14	1083888420	886544846	1970433266	55	2052470093	2379931641	4432401734	46.3
15	979301131	991132135	1970433266	49.69	2237514358	2194887376	4432401734	50.48
16	1015319573	1081236081	2096555654	48.42	2209784778	2096494568	4306279346	51.31
17	1035277679	1061277975	2096555654	49.37	2169707805	2136571541	4306279346	50.38
18	1072345116	1069003709	2141348825	50.07	2127098206	2134387969	4261486175	49.91

19	1083008126	1058340699	2141348825	50.57	2107199802	2154286373	4261486175	49.44
20	1068669506	1072679319	2141348825	49.9	2130918084	2130568091	4261486175	50
21	1114085103	1095388782	2209473885	50.42	2079044022	2114317093	4193361115	49.57
22	1106560886	1102912999	2209473885	50.08	2090302131	2103058984	4193361115	49.84
23	1110419363	1099054522	2209473885	50.25	2090751921	2102609194	4193361115	49.85
24	1150613143	1141891433	2292504576	50.19	2050398491	2059931933	4110330424	49.88
25	1147427588	1145076988	2292504576	50.05	2053742476	2056587948	4110330424	49.96
26	1147941766	1144562810	2292504576	50.07	2052871322	2057459102	4110330424	49.94
27	1204154994	1203020567	2407175561	50.02	1996970067	1998689372	3995659439	49.97
28	1203816053	1203359508	2407175561	50	1997427835	1998231604	3995659439	49.98
29	1227550009	1226736537	2454286546	50.01	1973660230	1974888224	3948548454	49.98
30	1227108369	1227178177	2454286546	49.99	1974351939	1974196515	3948548454	50
31	1227110080	1227176466	2454286546	49.99	1974327851	1974220603	3948548454	50
32	1271523744	1271812897	2543336641	49.99	1929903958	1929594401	3859498359	50
33	1271593007	1271743634	2543336641	49.99	1929872565	1929625794	3859498359	50
34	1271625565	1271711076	2543336641	49.99	1929768012	1929730347	3859498359	50
35	1351652483	1351659738	2703312221	49.99	1849770140	1849752639	3699522779	50
36	1351627230	1351684991	2703312221	49.99	1849785835	1849736944	3699522779	50
37	1395619940	1395679198	2791299138	49.99	1805788718	1805747144	3611535862	50
38	1395668625	1395630513	2791299138	50	1805755428	1805780434	3611535862	49.99
39	1395665096	1395634042	2791299138	50	1805754591	1805781271	3611535862	49.99

Everything is consistent !

6. **Test of the sequence**

When you check the "Statistics on n_1", which is necessary to test the sequence, we also have in the results a visualization and a modeling of the behavior of the lists having the same number of "type 1" transitions.
This statistical approach is less satisfactory than the one chosen in this document. It is providedt for reference only.

Even if it is completely useless, this program is designed for parallel execution by partitioning the calculations, even if they are performed on the same computer.

Choose the value of b_1 : [ ] OK (odd relative integer)

Sequence test for $v_0 < 2^N$: See/Hide code See/Hide code for results

N : 17 ◉ All tests ○ Random tests during 1 minute Try Stop Previous results
☐ Statistics on v_0 only (calculations are much more short, no validation of the sequence)
☑ Statistics on n_1 (necessary for the validation of the sequence)
☐ Adding statistics on n_2 (interesting to detect cycles if $b_1 \neq 1$)
☐ Adding transition statistics (for $n < N$)
☐ Adding statistics on r_N (longer calculations)
Found cycles ($v_0 = 0$ is eliminated and if b1 < 0 then v_0 = -b1 is also omitted) :

VIII. General case, in particular $5n+1$

1. Case a_3 odd

Thus, we consider the sequence defined with b_1 odd :

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \dfrac{a_3 v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

1. **Case $a_3 < 0$**

   We can reduce it to the case $a_3 > 0$ by also changing b_1 in its opposite

2. **Case $a_3 = 3$**

   1. **Case $|b_1| \leq 1024$**

      This case is already examined in detail in this document

   2. **Case $|b_1| > 1024$**

      There is no theoretical obstacle to proving similar theorem.

      We should first verify the sequence up to $2^{N_0+B_1}$.

      For $1024 < |b_1| < 2048$, we find 737 as the maximum length of one cycle for $b_1 = 1699$ and $v_0 = 23$. Then, the reasoning remains valid with $N_0 = 17$
      Even if it is necessary to increase N_0 which is initially equal to 17, it becomes unrealistic to verify the sequence for $|b_1| > 2^{51}$ with the current computer resources (in 2024), since we have only done a single verification for the standard Syracuse sequence up to 2^{68} (which, by the way, is useless for the proof of the theorem).

3. **Case $a_3 = 5$**

   The value of a_3 can only be changed here : ◉ 3 ○ 5 and the maximum number of steps for the study of convergence (possible divergence for $a_3 = 5$) : 450 ⌄
   Note : To indicate that $a_3 = 5$, the background color of the document is "light gray". All "Try" are operational, only the results are different.

   When trying to test the sequence, we realize that there seems to be a problem because with $b_1 = 1$, for $v_0 = 7$, we do not have a cycle after 10000 steps, which points towards a possible divergence.

   The maximum number of steps can be lowered to 450 after noticing that there was no cycle of length greater than 7 for $b_1 = 1$
   The value of b_1 doesn't really matter here, we will take $b_1 = 1$, because only the trivial cycles are different.

   We find these numbers of cases where there seems to be a divergence for $b_1 = 1$ and $N_0 = 20$:

Maximum number of transitions	Number of cases without convergence
50	523009
100	520747
200	519433
400	519296
425	519296
450	519293
500	519293
600	519293
800	519293
1000	519293

2000	519293

We will note that the number seems to be constant from a maximum number of transitions greater than 450.

This number is not very significant because we only perform $\frac{2^{N_0}}{2} = 2^{19} = 524288$ tests.

The construction of the JGL boundary is identical, with patterns. We notice that the pattern of length 339 seems to be the most suitable (even if it appears less than the pattern of 1054 for the case $a_3 = 3$)

The number of transition lists for cycles is not upper bounded by the number of Dyck's words because this time, it's less restrictive because $X = \frac{\text{Ln2}}{\text{Ln5} - \text{Ln2}}$ and $\frac{X}{1+X} \approx 0.43 < \frac{1}{2}$

However, we have :

- For $b_1 > 0$ and $\text{N} = 10790$, $Cy\ (\text{N}) = Cy\ (10790) \approx 2^{10790 - 170.51} = 2^{\text{N} - 170.51}$
- For $b_1 < 0$ and $\text{N} = 7804$, $Cy\ (\text{N}) = Cy\ (7804) \approx 2^{7804 - 128.27} = 2^{\text{N} - 128.27}$

It is enough to take $N_0 = 20$ to be assured of these respective minimum values of cycle length.

Here are the results of the $\frac{2^{20}}{2} = 2^{19} = 524288$ tests for $b_1 = 1$ and $N_0 = 20$:

For the first time : more than 2000 steps are required to verify the conjecture for 7
Continue without displaying other such values
For 519293 values, the conjecture is not verified.
Values are :
7,9,11,21,23,25,29,31,35,37,39,41,45,47,49,53,55,57,59,61,63,67,69,71,73,75,77,79,81,85,87,89,91,93,95,9

b_1	v_0	Length	Values
1	1	5	1; 3; 8; 4; 2
1	13	7	13; 33; 83; 208; 104; 52; 26
1	17	7	17; 43; 108; 54; 27; 68; 34

Then, to prove that there are no cycles other than the trivial cycles (for the values of $v_0 < 2^{N_0 + B_1} = 2^{20 + B_1}$ which converge), there would be no difficulties. It is enough to revisit the method for $a_3 = 3$, we assume the result.

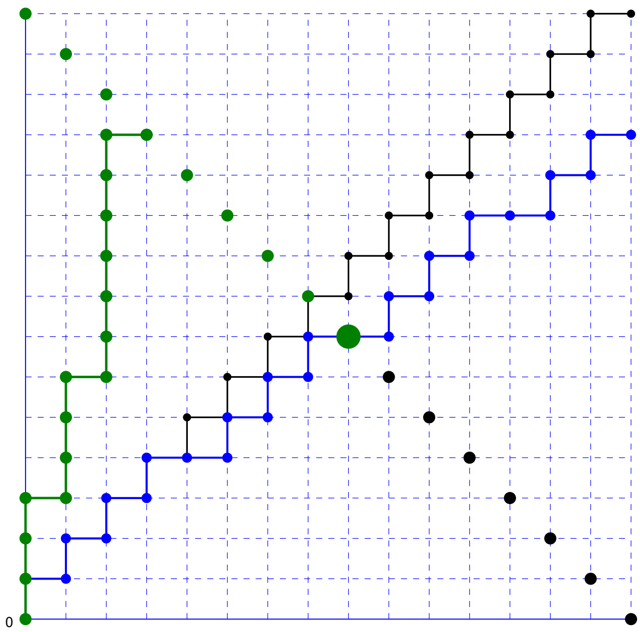
We are going to study the divergence towards infinity (for $b_1 = 1$).

To begin with, and as we had done for $a_3 = 3$, we have a single list of transitions such as a cycle is reached from a value lower than the minimum cycle value. This is $v_0 = 5$ and the corresponding transition list.

As for cycles, the constraint for a transition list of length N to be such that $v_n \geq v_0$ for all $n \leq \text{N}$ is weaker than for $a_3 = 3$ and even for the Catalan's triangle (JGL boundary under the diagonal).

In the diagram below :

- The blue line represents the boundary JGL
- The green line corresponds to a valid transition list because it is always above JGL
- The black line corresponds to the boundary for the Catalan's triangle
- Green dots are the endpoints of valid transition lists
- Black dots are the endpoints of transition lists that are not valid



We will notice that we always have (regardless of the value of a_3) :

- *Up* (N+1) = 2 *Up* (N) if t_N if of "type 0" in JGL (or JGL[N+1] = '0') for N > 0
- *Up* (N+1)= if t_N if of "type 1" in JGL (or JGL[N+1] = '1') for N > 0 with _EN(_N), the number of transition lists of length N, reaching exactly the JGL boundary in N

Then, as long as the boundary is "vertical" (t_N of "type 1"), *Up* (N) = 1 and then *Up* (N+1) > *Up* (N) in all cases, therefore *Up* (N) is an increasing function, as we might have suspected by looking at the possible paths in the previous diagram.

The previous program for calculating *Up* (N) is functional but this implementation is even more general. In the results p is such that *Up* (N) = 2^{N-p}

See/Hide code

Test for the maximum value of N : 5000 for a boundary slope of Ln(2)/Ln(5) Try

We get the following results with $a_3 = 5$ i.e. a "slope" equal to $\frac{\text{Ln2}}{\text{Ln5}}$ (not to be taken in the strict sense of that of a straight line, but as an increase in relation to the length of the path) :

N	*p*	*Up(N)*
1	1	$\sim 2^{0} \sim 2^{0\times N} \sim 2^{N-1}$
2	1	$\sim 2^{1} \sim 2^{0.5\times N} \sim 2^{N-1}$
3	1.415037	$\sim 2^{1.584963} \sim 2^{0.528321\times N} \sim 2^{N-1.415037}$
4	1.415037	$\sim 2^{2.584963} \sim 2^{0.646241\times N} \sim 2^{N-1.415037}$
5	1.678072	$\sim 2^{3.321928} \sim 2^{0.664386\times N} \sim 2^{N-1.678072}$
6	1.678072	$\sim 2^{4.321928} \sim 2^{0.720321\times N} \sim 2^{N-1.678072}$
7	1.870717	$\sim 2^{5.129283} \sim 2^{0.732755\times N} \sim 2^{N-1.870717}$
8	1.870717	$\sim 2^{6.129283} \sim 2^{0.76616\times N} \sim 2^{N-1.870717}$
9	1.870717	$\sim 2^{7.129283} \sim 2^{0.792143\times N} \sim 2^{N-1.870717}$
10	1.944718	$\sim 2^{8.055282} \sim 2^{0.805528\times N} \sim 2^{N-1.944718}$
20	2.170575	$\sim 2^{17.829425} \sim 2^{0.891471\times N} \sim 2^{N-2.170575}$
30	2.2812	$\sim 2^{27.7188} \sim 2^{0.92396\times N} \sim 2^{N-2.2812}$
40	2.349178	$\sim 2^{37.650822} \sim 2^{0.941271\times N} \sim 2^{N-2.349178}$

50	2.387105	$\sim 2^{47.612895} \sim 2^{0.952258\times N} \sim 2^{N-2.387105}$
60	2.410317	$\sim 2^{57.589683} \sim 2^{0.959828\times N} \sim 2^{N-2.410317}$
70	2.433388	$\sim 2^{67.566612} \sim 2^{0.965237\times N} \sim 2^{N-2.433388}$
80	2.448684	$\sim 2^{77.551316} \sim 2^{0.969391\times N} \sim 2^{N-2.448684}$
90	2.45861	$\sim 2^{87.54139} \sim 2^{0.972682\times N} \sim 2^{N-2.45861}$
100	2.469113	$\sim 2^{97.530887} \sim 2^{0.975309\times N} \sim 2^{N-2.469113}$
200	2.499393	$\sim 2^{197.500607} \sim 2^{0.987503\times N} \sim 2^{N-2.499393}$
300	2.504566	$\sim 2^{297.495434} \sim 2^{0.991651\times N} \sim 2^{N-2.504566}$
400	2.505715	$\sim 2^{397.494285} \sim 2^{0.993736\times N} \sim 2^{N-2.505715}$
500	2.506021	$\sim 2^{497.493979} \sim 2^{0.994988\times N} \sim 2^{N-2.506021}$
600	2.506105	$\sim 2^{597.493895} \sim 2^{0.995823\times N} \sim 2^{N-2.506105}$
700	2.50613	$\sim 2^{697.49387} \sim 2^{0.99642\times N} \sim 2^{N-2.50613}$
800	2.506138	$\sim 2^{797.493862} \sim 2^{0.996867\times N} \sim 2^{N-2.506138}$
900	2.50614	$\sim 2^{897.49386} \sim 2^{0.997215\times N} \sim 2^{N-2.50614}$
1000	2.506141	$\sim 2^{997.493859} \sim 2^{0.997494\times N} \sim 2^{N-2.506141}$
2000	2.506141	$\sim 2^{1997.493859} \sim 2^{0.998747\times N} \sim 2^{N-2.506141}$
3000	2.506141	$\sim 2^{2997.493859} \sim 2^{0.999165\times N} \sim 2^{N-2.506141}$
4000	2.506141	$\sim 2^{3997.493859} \sim 2^{0.999373\times N} \sim 2^{N-2.506141}$
5000	2.506141	$\sim 2^{4997.493859} \sim 2^{0.999499\times N} \sim 2^{N-2.506141}$
6000	2.506141	$\sim 2^{5997.493859} \sim 2^{0.999582\times N} \sim 2^{N-2.506141}$
7000	2.506141	$\sim 2^{6997.493859} \sim 2^{0.999642\times N} \sim 2^{N-2.506141}$
8000	2.506141	$\sim 2^{7997.493859} \sim 2^{0.999687\times N} \sim 2^{N-2.506141}$
9000	2.506141	$\sim 2^{8997.493859} \sim 2^{0.999722\times N} \sim 2^{N-2.506141}$
10000	2.506141	$\sim 2^{9997.493859} \sim 2^{0.999749\times N} \sim 2^{N-2.506141}$
20000	2.506141	$\sim 2^{19997.493859} \sim 2^{0.999875\times N} \sim 2^{N-2.506141}$
30000	2.506141	$\sim 2^{29997.493859} \sim 2^{0.999916\times N} \sim 2^{N-2.506141}$
40000	2.506141	$\sim 2^{39997.493859} \sim 2^{0.999937\times N} \sim 2^{N-2.506141}$
50000	2.506141	$\sim 2^{49997.493859} \sim 2^{0.99995\times N} \sim 2^{N-2.506141}$

Remember that p is such that $Up\ (\mathrm{N}) = 2^{\mathrm{N}-p}$

It is of course impossible to draw conclusions from just a few values, but it would seem that p diverges when the "slope" of the boundary is greater than 0.50 and p converges when the "slope" is less than 0.48 (by considering the valid paths as those "above" the boundary)

Even though there are many variants of the Catalan's triangle, I neither have sufficient mathematical knowledge nor any reference books, so I do not know if the case of any boundary has already been studied to have a formal expression (if it exists) of the quantity we are looking for, sum of the numbers of possibilities to reach locations on the opposite diagonal (points in the previous diagram).

Let's try to find a simpler method of minoration of $Up\ (\mathrm{N})$ to prove that there exists l such that $Up\ (\mathrm{N}) > 2^{\mathrm{N}-l}$ (for all N sufficiently large) because then it will be easy to conclude for divergence due to the distribution of values v_0.

According to the results of the previous tests, if we have a convergence for a "slope" less than 0.48 and that $\frac{\mathrm{Ln}2}{\mathrm{Ln}5} \approx 0.43$, this means that we have a comfortable margin to choose a boundary for which we can minor more easily $Up\ (\mathrm{N})$.

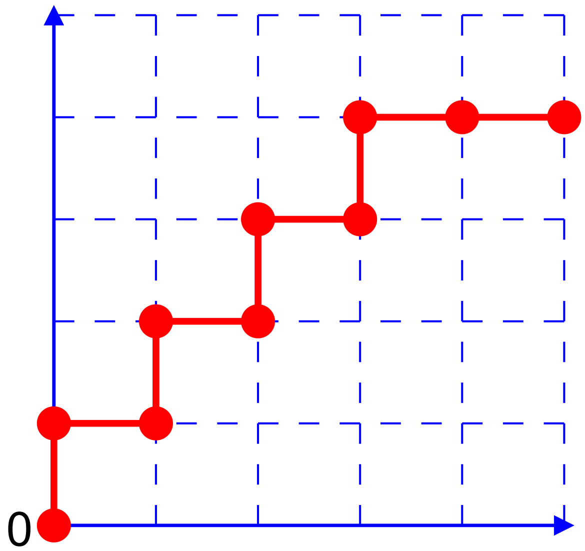
The disadvantage of $\frac{\text{Ln}2}{\text{Ln}5}$ is that it is not rational and even if there are patterns in the boundary, they do not repeat themselves rigorously identically in sequence.

The idea is to approximate the boundary by a repetitive pattern.

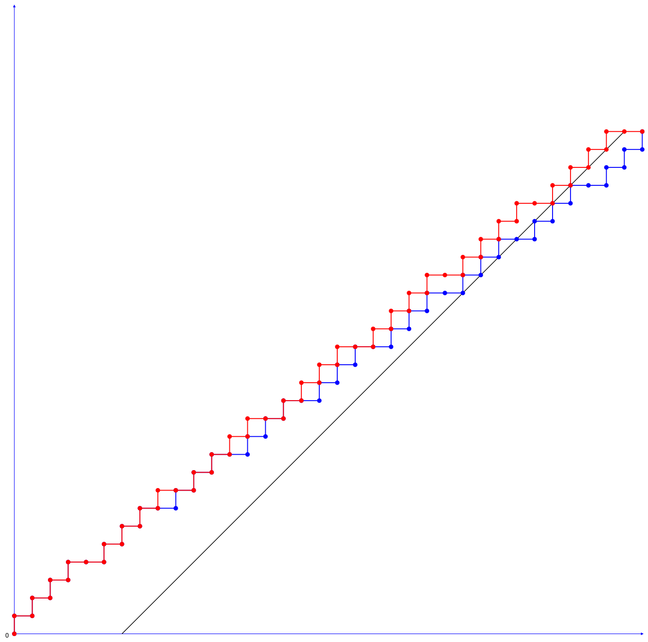
To obtain a lower bound for the number of paths (or transition lists), it is enough to take an approximation by excess of $\frac{\text{Ln}2}{\text{Ln}5}$.

This was done in the previous sections (see paragraph IV-9 if we have properly checked $a_3 = 5$ at the beginning of this paragraph, light gray background for the document).

We find the "approximation" $\frac{m}{\text{N}} = \frac{1}{2} = 0.5$ which is too large because it is greater than 0.48, the next is $\frac{4}{9}$ $\approx 0.444...$ which is suitable. The advantage is that the pattern is small (rectangle 5 by 4) and that we will be able to perform the exact calculations easily.



This gives this representation with the true JGL boundary in blue and the one with the pattern in red (the black line will be used for the Catalan's trapezoids) :



Let U_N the number of transition lists of length N above the boundary with the previous pattern (rectangle 4×5 or "slope" of boundary $\frac{4}{9}$).

Let's be interested in the extracted sequence, U_{9k}, which corresponds to the pattern

If, for $k > k_0$, we have $U_{9(k+1)} > 2^{9-e_k}U_{9k}$ and $U_{9k_0} = 2^{9k_0 - l_0}$

Then for $t > k_0$, $U_{9t} > \prod_{k=k_0}^{t-1} 2^{9-e_k}U_{k_0} = 2^{\left[9(t-k_0) - \sum_{k=k_0}^{t-1} e_k\right]}U_{9k_0} = 2^{\left[9(t-k_0) - \sum_{k=k_0}^{t-1} e_k\right]}\times 2^{9k_0 - l_0} = 2^{9t - g(t)}$ with

$g(t) = l_0 + \sum_{k=k_0}^{t-1} e_k$

If $g(t)$ had a limit l when t tends to infinity, then we would have $U_{9k} > 2^{9k-l}$ for $k > k_0$ large enough

On the other hand, we have $U_N < Up(N)$, then for $k > k_0$ large enough, $Up(9k) > 2^{9k-l}$

Let's express $U_{9(k+1)}$ from U_{9k} :

To simplify, we note $A_{k,n}$, the number of paths on the boundary (with the pattern) at the point with coordinates $(5k+n, 4k+n)$
With the pattern, we have the following very simple relations :

$U_{9k+1} = 2U_{9k} - A_{k,0}$
$U_{9k+2} = 2U_{9k+1} = 2^2U_{9k} - 2A_{k,0}$
$U_{9k+3} = 2U_{9k+2} - A_{k,1} = 2^3U_{9k} - 4A_{k,0} - A_{k,1}$
$U_{9k+4} = 2U_{9k+3} = 2^4U_{9k} - 8A_{k,0} - 2A_{k,1}$
$U_{9k+5} = 2U_{9k+4} - A_{k,2} = 2^5U_{9k} - 16A_{k,0} - 4A_{k,1} - A_{k,2}$
$U_{9k+6} = 2U_{9k+5} = 2^6U_{9k} - 32A_{k,0} - 8A_{k,1} - 2A_{k,2}$
$U_{9k+7} = 2U_{9k+6} - A_{k,3} = 2^7U_{9k} - 64A_{k,0} - 16A_{k,1} - 4A_{k,2} - A_{k,3}$
$U_{9k+8} = 2U_{9k+7} = 2^8U_{9k} - 128A_{k,0} - 32A_{k,1} - 8A_{k,2} - 2A_{k,3}$
$U_{9(k+1)} = U_{9k+9} = 2U_{9k+8} = 2^9U_{9k} - 256A_{k,0} - 64A_{k,1} - 16A_{k,2} - 4A_{k,3}$
As $A_{k,0} \le A_{k,1} \le A_{k,2} \le A_{k,3}$ then we have a very large lower bound

$$\mathbf{U_{9(k+1)} \ge 2^9U_{9k} - (256+64+16+4)A_{k,3} = 2^9U_{9k} - 340A_{k,3}}$$

Let's express the quantity e_k by setting $2^9U_{9k} - 340A_{k,3} = 2^{9-e_k}U_{9k}$
Therefore, $2^9U_{9k}\left(1 - \frac{340A_{k,3}}{2^9U_{9k}}\right) = 2^9U_{9k}\times2^{-e_k}$

$\Leftrightarrow 1 - \frac{340A_{k,3}}{2^9U_{9k}} = 2^{-e_k}$

$\Leftrightarrow \mathrm{Ln}\left(1 - \frac{340A_{k,3}}{2^9U_{9k}}\right) = -e_k\mathrm{Ln}2$ si $\frac{340A_{k,3}}{2^9U_{9k}} < 1$ which will be justified later

By using the asymptotic equivalent $\mathrm{Ln}(1+u) = u$ when u tends to 0

We have, if $\frac{340A_{k,3}}{2^9U_{9k}}$ tends to 0, which will be justified later :

$$\mathbf{e_k = \frac{340\, A_{k,3}}{Ln2 \times 2^9 U_{9k}} \approx \frac{0.958\, A_{k,3}}{U_{9k}} < \frac{A_{k,3}}{U_{9k}}}$$

We aim to upper bound e_k so we first upper bound $A_{k,3}$ then lower bound U_{9k}

Let's upper bound $A_{k,3}$:

$A_{k,3}$ is also the number of paths at the point with coordinates $(5k+n+1, 4k+n)$, since by increasing the length of 1, we can only move upward or to the right (to stay above the boundary). Therefore, we must have come from the left, from the point $(5k+n, 4k+n)$, which corresponds to $A_{k,3}$

Using the Catalan's trapezoids formula[6], as the boundary of the pattern is above the diagonal passing through $(5k+n+1, 4k+n)$ (black line in the previous diagram), we can lower bound $A_{k,3}$ by :

$A_{k,3} < C_{9k+6}^{5k+3} - C_{9k+6}^{5k+3-(k+1)} = C_{9k+6}^{5k+3} - C_{9k+6}^{4k+2}$

In fact, certainly due to the shape of the pattern (which differs from the diagonal only at the last step), it would seem that, although this is not a proof (even if we might be able to adapt it to account for this additional step) :

$A_{k,3} = \frac{1}{k+1}(C_{9k+6}^{5k+3} - C_{9k+6}^{4k+2})$, the first values being $A_{0,3} = 5$, $A_{1,3} = 715$, $A_{2,3} = 178296$, equality is verified for $k \leq 1000$

And anyway, let's take as a very broad upper bound (from the trapezoids formula), by neglecting the second term, which is indeed very large :

$A_{k,3} < C_{9k+6}^{5k+3} = \frac{(9k+6)!}{(4k+3)!(5k+3)!}$ and let's calculate an asymptotic equivalent of C_{9k+6}^{5k+3} for k large enough with the approximation $\mathrm{Ln}(n!) \approx n\,\mathrm{Ln}(n) - n$.

$\mathrm{Ln}(A_{k,3}) < (9k+6)\mathrm{Ln}(9k+6) - (9k+6) - (4k+3)\mathrm{Ln}(4k+3) + 4k+3 - (5k+3)\mathrm{Ln}(5k+3) + 5k+3$ for $k \geq 100$

$\mathrm{Ln}(A_{k,3}) < (9k+6)\mathrm{Ln}(9k+6) - (4k+3)\mathrm{Ln}(4k+3) - (5k+3)\mathrm{Ln}(5k+3)$ after simplifications

By using $\mathrm{Ln}(9k+6) \approx \mathrm{Ln}(9k)$, $\mathrm{Ln}(4k+3) \approx \mathrm{Ln}(4k)$ and $\mathrm{Ln}(5k+3) \approx \mathrm{Ln}(5k)$, then

$\mathrm{Ln}(ab) = \mathrm{Ln}(a) + \mathrm{Ln}(b)$

$\mathrm{Ln}(A_{k,3}) < (9k+6)\mathrm{Ln}(9k) - (4k+3)\mathrm{Ln}(4k) - (5k+3)\mathrm{Ln}(5k)$

$\mathrm{Ln}(A_{k,3}) < (9k+6)\mathrm{Ln}(9) - (4k+3)\mathrm{Ln}(4) - (5k+3)\mathrm{Ln}(5)$ after simplifications

$\mathrm{Ln}(A_{k,3}) < (9\,\mathrm{Ln}(9) - 4\,\mathrm{Ln}(4) - 5\,\mathrm{Ln}(5))k + 6\,\mathrm{Ln}(9) - 3\,\mathrm{Ln}(4) - 3\,\mathrm{Ln}(5) \approx 6.183k + 4.2$

$\mathbf{A_{k,3} < e^{6.183k+4.2} = 2^{(6.183k+4.2)/Ln2} = 2^{8.921k+6.06}}$

Let's lower bound U_{9k} :

The boundary with the pattern is below the diagonal (because $\frac{4}{9} < \frac{1}{2}$) then U_{9k} is greater than the number of paths above the diagonal (I really believe that I am reversing the customary conventions, but it is related to $\frac{3^m}{2^N} > 1$)

By applying the same reasoning as in paragraph VII-1, it is easy to find that $U_{9k} > C_{9k}^{9k/2}$ for k even.

By applying the Stirling formula for $n! \approx \sqrt{2\pi n}\,(\frac{n}{e})^n$, we find :

$$U_{9k} > \frac{(9k)!}{(\frac{9k}{2})!^2} \approx \frac{\sqrt{2\pi\times 9k}\,(\frac{9k}{e})^{9k}}{\pi\times 9k\,(\frac{9k}{2e})^{9k}} = \frac{\sqrt{2}}{\sqrt{9\pi k}}\times 2^{9k} = \frac{\sqrt{2}}{\sqrt{9\pi}}\times 2^{9k-\mathrm{Ln}(k)/(2\mathrm{Ln}2)}$$

$$\mathbf{U_{9k} > 2^{9k-\mathrm{Ln}(k)/(2\mathrm{Ln}2)}}$$

In conclusion, we can return to e_k because the assumptions are justified for the asymptotic equivalent of $\mathrm{Ln}(1+u) = u$:

$\mathbf{e_k < \frac{A_{k,3}}{U_{9k}} < 2^{8.921k+6.06-9k+\mathrm{Ln}(k)/(2\mathrm{Ln}2)} \approx 2^{-0.079k}}$, geometric sequence with common ratio $2^{-0.079} = 0.947 < 1$ therefore the series converges and the function $g(9k)$ has a limit.

To summarize, by taking $k_0 = 1000$, we have $l_0 = 2.69258223$ and $\sum_{k=k_0}^{+\infty} e_k = \frac{e_{k_0}}{1-q} = \frac{2^{-79}}{1-0.947} \approx 0.$

Therefore $Up\ (9t) > U_{9t} > 2^{9t-l_0} = 2^{9t-2.693}$, which is consistent with the results table that suggests a value close to $l = 2.506141$

Given the previous relations that link U_{9k+i} to U_{9k} for $0 < i < 9$, which do not affect the 10th decimal place of l for $k > 1000$, then $U_N > 2^{N-2.7}$ for all $N > 9000$ (we can verify that this is also true for $N \le 9000$).

And finally : $\boldsymbol{Up\ (N) > 2^{N-2.7}}$ **for all N**.

We will use the previous method (as in the case $a_3 = 3$, previously detailed in paragraph VI-4-3) regarding the distribution of v_0 solutions of the transition lists, but this time by lower bounding the number of remaining lists. It is actually while trying to test the robustness of the method that the case $a_3 = 5$ was considered.

At first glance, by simplifying the process, by dividing by 2 each time, we could have the minimum value of v_0 on 3 bits, like $v_0 = 7$.

More rigorously, by setting $v_0 = \sum_{n=0}^{N-1} d_n \times 2^n < 2^N$:

We can see $d_{N-1} = 1$ as a Bernouilli trial of parameter $p = \frac{1}{2}$, optimal parameter for the application of statistical results.

For the last transition :

If we consider X_n, the number of times $d_{N-1} = 1$ (for the transition $N-1$) for all the $n = Up\ (N)$ transition lists, it is then a binomial distribution and $Z_n = \frac{X_n - \frac{n}{2}}{\sqrt{p(1-p)n}} = \frac{X_n - \frac{n}{2}}{\sqrt{\frac{1}{2}(1-\frac{1}{2})n}} = \frac{X_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$ converges towards the standard normal distribution (for $n > 30$ because $p = \frac{1}{2}$).

Moreover, we have $P(Z_n \ge -4) = P(Z_n \le 4) > 0.9999$ (since in the tables of the standard normal distribution, the value is 1.0000)

We can consider that we are in this case to lower bound the remaining number of v_0, i.e. the number of

$v_0 < 2^{N-1}$ for the candidate transition lists. If that were not the case, it is not a problem since we will apply the same reasoning for the previous transitions.

However $Z_n \geq -4 \Leftrightarrow X_n > \frac{n}{2} - 4\sqrt{\frac{n}{4}} = \frac{n}{2} - 2\sqrt{n} = \frac{n}{2}(1 - 4n^{-1/2})$

Let A_1 this lower bound of X_n and $A_0 = n$ the initial number of candidate lists.

If $n > 2^{10}$ then we can always use the convergence towards the standard normal distribution (because the limit of use is close to $n > 2^5$) and $u = 4n^{-1/2} < 2^{-3}$

The inequality $1 - u < e^{-u}$ does not allow for a lower bound but $1 - u > 1 - \frac{3}{2}u$.

We could lower bound $1 - u$ by $e^{-3u/2}$ if $f(u) = 1 - u - e^{-3u/2} > 0$ for $0 < u < 2^{-3}$, the interval that interests us.

Let's quickly examine the sign of f :

$f'(u) = -1 + \frac{3}{2}e^{-3u/2} \geq 0 \Leftrightarrow e^{-3u/2} \geq \frac{2}{3} \Leftrightarrow \frac{-3}{2}u \geq \mathrm{Ln}(\frac{2}{3}) \Leftrightarrow u \leq -\frac{2}{3}(\mathrm{Ln}2 - \mathrm{Ln}3) \approx 0.27 > 2^{-3}$

As $f(0) = 0$, $f(u) \geq 0$ on $[\ 0 \quad 2^{-3}\ ]$ and then, we can use the loxer bound :

$1 - u > e^{-3u/2}$

$A_1 = \frac{n}{2}(1 - u) > \frac{n}{2}e^{-3u/2} = \frac{n}{2}e^{-4n^{-1/2} \times 3/2} = \frac{n}{2}e^{-6n^{-1/2}} = R_1$, the lower bound by using the exponential, which allows for a more straightforward chaining

We can apply exactly the same reasoning with the transition $N - 2$ with the same formalism, as long as the number of remaining lists is sufficient.

We have $R_2 \geq \frac{R_1}{2}e^{-6R_1^{-1/2}}$

However $R_1^{-1/2} = (\frac{n}{2}e^{-6n^{-1/2}})^{-1/2} = (\frac{n}{2})^{-1/2}e^{3n^{-1/2}} > (\frac{n}{2})^{-1/2}$ because $3n^{-1/2} > 0$

Therefore $R_2 \geq \frac{n}{4}e^{-6n^{-1/2}}e^{-6(n/2)^{-1/2}} = \frac{n}{4}e^{-6n^{-1/2}}e^{-6n^{-1/2}\sqrt{2}} = \frac{n}{4}e^{-6n^{-1/2}(1+\sqrt{2})}$

Starting again with the same formalism for d_{N-k}, as long as we can use convergence towards the standard normal distribution, we have :

$R_k \geq \frac{n}{2^k}e^{-6n^{-1/2}S_k}$ with $S_k = \sum_{i=1}^{k}\sqrt{2}^{(i-1)} = \frac{\sqrt{2}^k - 1}{\sqrt{2} - 1} < \frac{\sqrt{2}^k}{\sqrt{2} - 1} = \frac{\sqrt{2^k}}{\sqrt{2} - 1}$

This result can be proven by induction, the formula being verified for $k = 1$ and $k = 2$.

If we assume that it is true for k, we prove that it is true for $k + 1$

$R_{k+1} \geq \frac{R_k}{2}e^{-6R_k^{-1/2}}$

However $R_k^{-1/2} = \left(\frac{n}{2^k}e^{-6n^{-1/2}S_k}\right)^{-1/2} = \left(\frac{n}{2^k}\right)^{-1/2}e^{3n^{-1/2}S_k} > \left(\frac{n}{2^k}\right)^{-1/2} = n^{-1/2}\sqrt{2}^k$ because $3n^{-1/2}S_k > 0$

Therefore $R_{k+1} \geq \frac{n}{2^{k+1}}e^{-6n^{-1/2}S_k}e^{-6n^{-1/2}\sqrt{2}^k} = \frac{n}{2^{k+1}}e^{-6n^{-1/2}S_{k+1}}$, the property is verified

We can slightly improve the notation of R_k with the last upper bound in S_k

$R_k \geq \frac{n}{2^k}e^{-6\sqrt{2^k}/((\sqrt{2}-1)\sqrt{n})}$

Let K such as $2^{K+17} \leq n \leq 2^{K+18}$, we lower bound n by a power of 2.

We are assured that $R_K > 2^{10}$, so we can use the formula related to convergence towards the standard normal distribution.

$$R_K \geq \frac{n}{2^K} e^{-6\sqrt{2^K}/((\sqrt{2}-1)\sqrt{2^{K+17}})} = \frac{n}{2^K} e^{-6/(2^8\sqrt{2}(\sqrt{2}-1))} = \frac{n}{2^K} e^{-3/(2^7(2-\sqrt{2}))} \approx$$
$$0.96 \times \frac{n}{2^K} > \frac{1}{2} \times \frac{n}{2^K} > \frac{2^{K+17}}{2^{K+1}} = 2^{16} = 65536$$

That is to say, when taking into account the K last transitions, the maximum relative error is 4% compared to the approximated value $\frac{n}{2^K}$, which means that the error is less than one "step" !

We can say, by analyzing the K last transitions, that the number of values of $v_0 < 2^{N-K}$ is greater than 65536.

But we have just proven that, for all N, Up (N) $= n = 2^{N-l} \approx 2^{N-2.7} > 2^{N-3}$ in case $a_3 = 5$
As we set K such that : $2^{K+17} \leq n \leq 2^{K+18}$
We therefore have, $K + 17 = N - 3$ and then $N - K = 20$, which means that the number of solutions $v_0 < 2^{20}$ is greater than $65536 = 2^{16}$ for all N.

For $N \geq 20$, let E_N, the set of minimum solutions $v_0 < 2^{20} \leq 2^N$ for all transition lists of length N such that $0 < n \leq N$, we have $v_n \geq v_0$ (because we are in the case $b_1 = 1$ or $v_0 > 0$)
The cardinality of E_N is greater than 2^{16} and less than 2^N
For all $N > P$, we have $E_N \subset E_P$, simply because the condition is more restrictive for $N > P$.

We consider $N_1 \geq 20$, fixed, for example $N_1 = 20$

Let's prove by contradiction that there exists at least one value $v_0 \in E_{N_1}$, for which there are an infinity of values of N for which $v_0 \in E_N$:
Assume that for any element e of E_{N_1}, e is in a finite number of sets E_N for distinct values of N (which is equivalent to saying that e reaches a trivial cycle), then, there exists M_e, the maximum of N such that $e \in E_N$. Since the cardinality of E_{N_1} is finite (less than 2^{20}), then it exists M_{E_1}, the maximum of all M_e for e being in E_{N_1}
If we consider $N = M_{E_1} + 1$, then, for each value e of E_{N_1}, we have $v_{M_{E_1}+1} < e$, and therefore $e \notin E_{M_{E_1}+1}$.
Yet, since we necessarily have $E_{M_{E_1}+1} \subset E_{N_1}$, then $E_{M_{E_1}+1} = \emptyset$, which contradicts the fact that the cardinality of $E_{M_{E_1}+1}$ is necessarily greater than 2^{16}, which concludes the proof.

Let $v_0 \in E_{N_1}$, such that v_0 is in an infinity of sets E_N, then we have no cycle from this value v_0 (to be precise, it would be necessary to remove the few values of v_0 that reach a trivial cycle with v_0 less than the minimum value of the trivial cycle. These cases are rare and there are none in the case $b_1 = 1$) and therefore for all N such that $v_0 \in E_N$, v_N is different from the values v_n for $0 \leq n < N$ and such that $v_0 \in E_n$
Let's prove by contradiction that v_n diverge to infinity :

Assume the opposite, then there would exist an upper bound M such that, for all n, $v_n < M$. But, if we consider the infinite set of values of N such that $v_0 \in E_N$, as the values of v_N are distinct (no cycle from v_0), then the finite number of possible values $M - v_0$ gives the contradiction.
Perhaps the value $v_0 = 7$ is one of them ?

> **For $a_3 = 5$ and $b_1 = 1$, there exists at least one value v_0 from which the sequence v (u too) diverges**

We can prove the same result for any value of b_1, only trivial cycles change.

4. **Case $a_3 > 5$**

If we assume that the sequence diverges for a value v_0 with $a_3 = 5$ and $b_1 = 1$, then there will necessarily be a divergence for $a_3 > 5$, the boundary being even less restrictive for the transition lists allowing for a possible divergence.

2. **Case a_3 even**

Thus, we consider the sequence defined as follows with b_1 even :

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \frac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \frac{a_3 v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

1. **Case $a_3 < 0$**

We can reduce to the case $a_3 > 0$ by also changing b_1 to its opposite

2. **Case $a_3 = 2$ or $a_3 = 4$ or $a_3 = 6$**

Then $a_3 = 2a$ with $1 \le a \le 3$

As $b_1 = 2k$ is even, then the transition of "type 1" becomes $v_{n+1} = a\, v_n + k$

if v_n is odd (transition of type 1)

The growth is at most equal to that of the standard Syracuse sequence, then we should have the same type of theorem with "trivial cycles" and no divergence.

3. **Case $a_3 = 8$**

As $b_1 = 2k$ is even, then the transition of "type 1"" becomes $v_{n+1} = 4\, v_n + k$

if v_n is odd (transition of type 1)

The JGL boundary would be the "diagonal" (because the proportion of "type 1" transitions is $\frac{\text{Ln}2}{\text{Ln}(4)} = \frac{1}{2}$), which we used in the case of the standard Syracuse sequence for the number of transition lists *Cy* (N) and *Up* (N), then we should have the same type of theorem with "trivial cycles" and no divergence.

4. **Case $a_3 > 8$**

By setting $a_3 = 2a$ with $a \ge 5$ and $b_1 = 2k$ is even,

then the transition of "type 1"" becomes $v_{n+1} = a\, v_n + k$ if v_n is odd (transition of type 1)

If we assume divergence for the case $a_3 = 5$, previously algorithmically explored, there would be values that diverge.

IX. **Appendices**

1. **"Theorem 1"**

Let u the Syracuse sequence defined by :

$$\begin{cases} u_0 > 0 \\ u_{n+1} = \frac{u_n}{2} \text{ if } u_n \text{ is even} \\ u_{n+1} = 3\, u_n + 1 \text{ if } u_n \text{ is odd} \end{cases}$$

Statement of Theorem 1 :

$\forall \varepsilon > 0, \exists (u_0, \mathrm{N}) \in \mathbb{N}^2$, with $u_0 > 4$ such as $\left| \frac{u_N}{u_0} - 1 \right| < \varepsilon$

In other words, the relative difference between u_0 and u_N can be as small as you want... and we have "almost" a non-trivial cycle.

It would suffice to take $u_0 = 1$ and N = 3 or any other value corresponding to the trivial cycle, this is why we add $u_0 > 4$

We can call these numbers u_0 "revolutionary" in that they nearly return to their starting point, $u_0 \approx u_N$, they made a revolution after a very chaotic journey because at each transition, they are multiplied by about 3 or divided by 2 which can represent gigantic absolute variations.

I) Notations and definitions :

First, let's define some elements and specify some notations

a) The reduced sequence of Syracuse : *v*

As if u_n is odd, u_{n+1} is even by construction, it is interesting to perform the following transition directly.
The reduced Syracuse sequence groups this transition.

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \frac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \frac{3v_n + 1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases}$$

We will use the sequence v in the proof because if we prove "Theorem 1" for v then we will have proven it for the standard Syracuse sequence u

b) $\mathrm{N} = m + d$

We will set $\mathrm{N} = m + d$ where m represents the number of "type 1" transitions and d represents the number of "type 0" transitions

c) The value $X = \frac{\mathrm{Ln}2}{\mathrm{Ln}3 - \mathrm{Ln}2}$, with Ln the natural logarithm function

II) Proof

1) Idea of the proof :

We consider $s_0(\mathrm{N}) = 2^d(2^m - 1)$ with $\mathrm{N} = m + d$ which trajectory can be easily determined by v.
We prove that $s_0(\mathrm{N})$ is suitable for particular pairs (m, d).

2) Study of the trajectory of $v_0(m) = 2^m - 1$ with $m > 0$:

We prove by induction that : For all n such as $0 \le n \le m$, $v_n(m)$ is odd and $v_m(m) = 3^m - 1$
This means that the first m transitions t_n are of "type 1" and the transition $m+1$ is of "type 0".
a) Let's verify the property for $m = 1$
$v_0(1) = 2^1 - 1 = 1$ is odd so the transition t_0 is of "type 1"

$$v_1(1) = \frac{3v_0(1)+1}{2} = \frac{3 \times 1 + 1}{2} = 2 = 3^1 - 1$$
so the property is true for $m = 1$
b) Assume the property is true for m and show that it is true for $m+1$

$v_0(m+1) = \sum_{i=0}^{m} 2^i = v_0(m) + 2^m = 2^m + v_0(m)$ is odd because 2^n is even for all $n > 0$

So, $v_1(m+1) = \dfrac{3v_0(m+1)+1}{2} = \dfrac{3(2^m + v_0(m))+1}{2} = \dfrac{3\times 2^m}{2} + \dfrac{3v_0(m)+1}{2} = 3\times 2^{m-1} + v_1(m)$ which is odd according to the property at rank m

By using the same reasoning, we obtain $v_2(m+1) = 3^2 \times 2^{m-2} + v_2(m)$ which is odd...

And finally $v_m(m+1) = 3^m + v_m(m) = 3^m + 3^m - 1 = 2\times 3^m - 1$ by replacing $v_m(m) = 3^m - 1$ according to the property at rank m

So $v_m(m+1)$ is odd and the transition $m+1$ is also of "type 1", it is the first part of the property at rank $m+1$

And $v_{m+1}(m+1) = \dfrac{3v_m(m+1)+1}{2} = \dfrac{3(2\times 3^m - 1)+1}{2} = \dfrac{2\times 3^{m+1}}{2} + \dfrac{-3+1}{2} = 3^{m+1} - 1$ which is the second part of the property at rank $m+1$ and this ends the proof by induction.

3) Let's prove that : If $s_0(\mathrm{N}) = 2^d(2^m - 1)$ with $\mathrm{N} = m + d$ then $s_{\mathrm{N}}(\mathrm{N}) = 3^m - 1$

The first d transitions are of "type 0" and eliminate the factor 2^d.

So $s_d(\mathrm{N}) = 2^m - 1 = v_0(m)$ and applying the previous property, the following m transitions are of "type 1" and $s_{\mathrm{N}}(\mathrm{N}) = v_m(m) = 3^m - 1$

4) Study of the limit of $\dfrac{s_{\mathrm{N}}(\mathrm{N})}{s_0(\mathrm{N})}$ when $\mathrm{N} \to +\infty$

$$\frac{s_{\mathrm{N}}(\mathrm{N})}{s_0(\mathrm{N})} = \frac{3^m - 1}{2^d(2^m - 1)} = \frac{3^m\left(1 - \dfrac{1}{3^m}\right)}{2^d \times 2^m\left(1 - \dfrac{1}{2^m}\right)} = \frac{3^m}{2^{m+d}} \times \frac{1 - \dfrac{1}{3^m}}{1 - \dfrac{1}{2^m}}$$

$$\lim_{m \to +\infty} \frac{s_{\mathrm{N}}(\mathrm{N})}{s_0(\mathrm{N})} = \lim_{m \to +\infty} \frac{3^m}{2^{m+d}}$$

The convergence is actually very rapid because by using a first-order Taylor expansion at 0 of $\dfrac{1}{1-u} = 1 + u$

We then have : $\left(1 - \dfrac{1}{3^m}\right)\left(1 + \dfrac{1}{2^m}\right) = 1 + \dfrac{1}{2^m} - \dfrac{1}{3^m} - \dfrac{1}{2\times 3^m} = 1 + \dfrac{1}{2^m}\left(1 - \left(\dfrac{2}{3}\right)^m - \dfrac{1}{3^m}\right)$

It is interesting to study the pairs (m, d) for which $\dfrac{3^m}{2^{m+d}} \approx 1$

5) Case where $\dfrac{3^m}{2^{m+d}} \approx 1$

$\dfrac{3^m}{2^{m+d}} \approx 1 \Leftrightarrow m\,\mathrm{Ln}3 - (m+d)\,\mathrm{Ln}2 \approx 0$ (with Ln the natural logarithm function)

$\Leftrightarrow \dfrac{m}{d} \approx \dfrac{\mathrm{Ln}2}{\mathrm{Ln}3 - \mathrm{Ln}2}$ (because $d \neq 0$)

Let's define $X = \dfrac{\mathrm{Ln}2}{\mathrm{Ln}3 - \mathrm{Ln}2} \approx 1{,}7095$

Conclusion : We will now focus on $\dfrac{m}{d} \approx X$

6) Approximation of X by fractions

X is an irrational number ($X \in \mathbb{R} \setminus \mathbb{Q}$).

It can be approximated as closely as desired by rational numbers.

Among all the approximations, we can focus on the fractions $\frac{m_p}{d_p}$ that correspond to the infinite continued fraction expansion of X because the convergence towards X is rapid :

m	d	N	m/d - x
1	1	2	-7.0951×10^{-1}
2	1	3	2.9049×10^{-1}
5	3	8	-4.2845×10^{-2}
12	7	19	4.7744×10^{-3}
41	24	65	-1.1780×10^{-3}
53	31	84	1.6613×10^{-4}
306	179	485	-1.4085×10^{-5}
665	389	1054	2.7677×10^{-7}
15601	9126	24727	-4.9171×10^{-9}
31867	18641	50508	9.6119×10^{-10}
79335	46408	125743	-1.9476×10^{-10}
111202	65049	176251	1.3650×10^{-10}
190537	111457	301994	-1.4274×10^{-12}
10590737	6195184	16785921	2.0821×10^{-14}
10781274	6306641	17087915	-4.7737×10^{-15}
53715833	31421748	85137581	2.7256×10^{-16}
171928773	100571885	272500658	-4.3877×10^{-17}
225644606	131993633	357638239	3.1454×10^{-17}
397573379	232565518	630138897	-1.1224×10^{-18}
6189245291	3620476403	9809721694	6.5206×10^{-20}

But $\frac{m_p}{d_p} - X = \frac{m_p}{d_p} - \frac{\text{Ln}2}{\text{Ln}3-\text{Ln}2} = \frac{m_p(\text{Ln}3-\text{Ln}2)-d_p\text{Ln}2}{d_p(\text{Ln}3-\text{Ln}2)} = \frac{m_p\text{Ln}3-(m_p+d_p)\text{Ln}2}{d_p(\text{Ln}3-\text{Ln}2)} = \frac{\text{Ln}\left(\frac{3^{m_p}}{2^{m_p+d_p}}\right)}{d_p(\text{Ln}3-\text{Ln}2)}$

That is : $\text{Ln}\left(\frac{3^{m_p}}{2^{m_p+d_p}}\right) = d_p(\text{Ln}3-\text{Ln}2)\left(\frac{m_p}{d_p}-X\right)$

By taking the exponential of both sides, we have :

(1) : $\frac{3^{m_p}}{2^{m_p+d_p}} = \text{e}^{(\text{Ln}3-\text{Ln}2)d_p(m_p/d_p-X)}$

7) Reminder of a general theorem on continued fractions and application

According to the properties of continued fractions (source Wikipedia), we have the following theorem :

If (h_n/k_n) have for limit l then : $\frac{1}{k_p(k_{p+1}+k_p)} < \left|l-\frac{h_p}{k_p}\right| < \frac{1}{k_p k_{p+1}}$ or $\frac{1}{k_{p+1}+k_p} < k_p\left|l-\frac{h_p}{k_p}\right| < \frac{1}{k_{p+1}}$

The previous theorem on continued fractions, applied in our case, gives :

$d_p\left|\frac{m_p}{d_p}-X\right| < \frac{1}{d_{p+1}}$ and then $\lim_{p\to+\infty} d_p\left|\frac{m_p}{d_p}-X\right| = 0$ because the sequence (d_p) is a strictly increasing, unbounded sequence, i.e. $\lim_{p\to+\infty} d_p = +\infty$

By using a Taylor expansion at 0 of $\text{e}^u = 1+u+o(u)$ in the expression (1), we have :

$$\lim_{p \to +\infty} \frac{3^{m_p}}{2^{m_p+d_p}} = \lim_{p \to +\infty} 1+(\text{Ln}3-\text{Ln}2)\,d_p\left(\frac{m_p}{d_p}-X\right)=1$$

In other words :$\forall \varepsilon > 0, \exists N_p \in \mathbb{N} : \left|\frac{3^{m_p}}{2^{m_p+d_p}}-1\right| < \varepsilon$

8) Conclusion :

For the reduced sequence v, by combining the results of 4) and 7), we can say with $v_0 = s_0(\text{N})$ and $\text{N} = m+d$:

By definition of the limit, $\forall \varepsilon > 0, \exists (v_0, \text{N}) \in \mathbb{N}^2$, with $v_0 > 4$ and $\text{N} > 3$ such as $\left|\frac{v_\text{N}}{v_0}-1\right| < \varepsilon$

We obviously have the result for the standard sequence of Syracuse u with $u_0 = s_0(N)$ and $\text{N} = 2m+d$ because there are m additional transitions :

$\forall \varepsilon > 0, \exists (u_0, \text{N}) \in \mathbb{N}^2$, with $u_0 > 4$ and $\text{N} > 3$ such that $\left|\frac{u_N}{u_0}-1\right| < \varepsilon$

9) Numerical results

NB : For $(m,d) = (1, 1)$, we indeed find the trivial cycle with $u_0 = 1$.

m	d	N	$\frac{m}{d}-X$	$v_0 = 2^d(2^m-1)$	$v_\text{N} = 3^m-1$	$\left\lvert\frac{v_\text{N}}{v_0}-1\right\rvert$	$\frac{1}{d_{p+1}}$
1	1	2	-7.0951×10^{-1}	2	2	0	5×10^{-1}
2	1	3	2.9049×10^{-1}	6	8	3.33×10^{-1}	3.33×10^{-1}
5	3	8	-4.2845×10^{-2}	248	242	2.42×10^{-2}	8.33×10^{-2}
12	7	19	4.7744×10^{-3}	524160	531440	1.39×10^{-2}	2.44×10^{-2}
41	24	65	-1.1780×10^{-3}	36893488147402326016	36472996377170786402	1.14×10^{-2}	3.23×10^{-2}
53	31	84	1.6613×10^{-4}	19342813113834064647815168	19383245667680019896796722	2.09×10^{-3}	5.59×10^{-3}
306	179	485	-1.4085×10^{-5}	$9.98959536101118\times10^{145}$	$9.97938882337109\times10^{145}$	1.02×10^{-3}	2.57×10^{-3}
665	389	1054	2.7677×10^{-7}	$1.93025830561934\times10^{317}$	$1.93034257116813\times10^{317}$	4.37×10^{-5}	1.10×10^{-4}

We indeed have a precision better than $\frac{1}{d_{p+1}}$.

The approximation $\left|\frac{v_\text{N}}{v_0}-1\right|$ corresponds well to $d\left|\frac{m}{d}-X\right|(\text{Ln}3-\text{Ln}2)$

The proof of this theorem is done in the standard case but it is generalized without any difficulty for any b_1.
We can even generalize this theorem by taking any positive odd coefficient a_3 instead of the restrictive case 3 by considering :

$$\begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{v_n}{2} \text{ if } v_n \text{ is even (transition of type 0)} \\ v_{n+1} = \dfrac{a_3 v_n + b_1}{2} \text{ if } v_n \text{ is odd (transition of type 1)} \end{cases} \quad \text{with } a_3 = 2a+1 \text{ and } a \ge 0$$

The statement of Theorem 1 becomes :

$\forall \varepsilon > 0, \exists (u_0, N) \in \mathbb{N}^2$, with $u_0 > maxSyr\_b1$ such as $\left| \dfrac{u_N}{u_0} - 1 \right| < \varepsilon$

- $u_0 > maxSyr\_b1$ to avoid trivial cycles
- The proof is done with $X = \dfrac{\text{Ln}2}{\text{Ln}(a_3) - \text{Ln}(2)}$

End of the proof of Theorem 1

2. **Jacques BALLASI's conjecture**

If we consider a transition list L of length N (different from that of the trivial cycle, i.e. 1010... and 01010...) and v_L, the only solution of the interval $[0, 2^N[$, the minimum solution that follows L.

Then, if we consider the set S_L of these N solutions for each list L_i, circular permutation of L, then the maximum of S_L is in the interval $[2^{N-1}, 2^N[$.

This conjecture has been verified for all lists for N ≤ 27 and also on a set of longer random lists.

But I don't have the formal proof, no reasoning by induction seems possible to me but I hope I am wrong! For example, if we take the list L = "001110011111", and L0 = L + "0" = "0011100111110", the lists for which $v_L \ge 2^{N-1}$ are not in those of L0, it is the same for L = "11011001100000010000000010" or L = "1100001111001001001101".

For $b_1 \neq 1$, we can formulate and "verify" an analogous conjecture by eliminating all circular permutations of lists containing the different cycles for this value of b_1

Testing my conjecture for a length of : 14 Rotate See/Hide code
Testing my conjecture for random lists during 10 secondes Random Tests

X. References

1. Syracuse conjecture in French, Wikipedia page : https://fr.wikipedia.org/wiki/Conjecture_de_Syracuse
2. Collatz's conjecture in English, Wikipedia page : https://en.wikipedia.org/wiki/Collatz_conjecture
3. French document by Shalom Eliahou, "Le problème 3n+1 : y-a-t-il des cycles non triviaux ?" who provides an excellent reasoning on the invariance of the product of the terms of a potential cycle after a transition. With the Farey intervals, he obtains that if the conjecture has been verified for $n < 5\times10^{18} \approx 2^{62.116}$, a cycle has at least a length of 17 billion and if the conjecture were verified up to 2.17×10^{20}, the minimum length would then be 186 billion (for the standard Syracuse sequence). As we have now verified the conjecture for $v_0 < 2^{68} \approx 2.95\times10^{20}$, the reasoning could be updated, and we are already certain that the minimum length of a non trivial cycle is at least 186 billion elements.
See the document in French : https://images.math.cnrs.fr/Le-probleme-3n-1-y-a-t-il-des-cycles-non-triviaux-III.html
4. Shalom Eliahou's English document, "The 3x + 1 problem : new lower bounds on nontrivial cycle lengths", this document, equivalent in English of the previous reference, is part of the references on the Wikipedia page of Collatz's conjecture in English
See the document in English : https://www.sciencedirect.com/science/article/pii/0012365X9390052U
5. Dyck's words and Catalan's triangle in French : https://fr.wikipedia.org/wiki/Triangle_de_Catalan

6. Catalan's triangle in English : https://en.wikipedia.org/wiki/Catalan%27s_triangle
7. Document of Eric Roosendaal, in which we have the list of "altitude flight" (glides) records.
   See the document in English : https:://www.ericr.nl/wondrous/glidrecs.html

XI. See also: Other documents

Remarks

- I am a French speaker and the English version of the document was obtained in "my" very approximate English or using the offline "Google Translate" appl.

  Knowing that online translation is certainly much better and considering the very rapid evolution of translation quality in general, it is certainly preferable to use online translation for the French part of the document.

- HTML documents were written with the formal editing tools for the web that I have developed in the early 2000s (perhaps even before the projects that gave rise to Mathjax), primarily for "Microsoft Internet Explorer".
  The rendering has not been updated for over 15 years and the characteristics of browsers have changed somewhat in the meantime, so there are a few inaccuracies in the placement of indices (which could be corrected)

  It is necessary to add approximately 500 kilobytes of the JTMath library and common style and image files to transform the native format.

  You need to download a subset of the JTMath libraries.

  You will then need to extract the files from the archive into the same directory as this document.

- The conversion from HTML format to PDF format to obtain a static document further degrades the quality (missing fraction lines) and increases the file size.

List of documents

- Proof of Theorem 0, version 1, different proof
  - PDF in French
    SHA256 : F9020212A09A4A95F14F1BF72E0F736E857C679C3E45EB90D500E65DFBF417A8
  - PDF in English
    SHA256 : B3FB1B0A47CCE34A818C3E7D1A617E242F302057D1577A6EC409AAE4B4AC6DFB
- Minimized function JBALLASI in Javascript (214 bytes)
  SHA256 : D5FF4B50E36CB783D9AA0C01FF38B06A68822456B88155C9264A0C1444740095
- ZIP archive of the subset of the JTMath library :
  SHA256 : D1FCC2A1A87178114201E7FA6D887B60BBE7DD536FB00B0C1550C7B7515CBCE6
- ZIP archive of symmetric encryptions programs with a secret key (in Javascript with NodeJS and in PHP)
  SHA256 : B1B25F57052EE856F0FE678702766B6DB3989EC2AD2B38211B18ECDCE900C0C2
- Version 3 : Proofs of theorems for the extensions of the Syracuse sequence :
  - DHTML + Javascript in French and English (without the minimized JTMath library)
    SHA256 : 5B81434FFC2A667AAAFB31D71179ACAF8FD1BF882220D5FB072580A047D17F71
  - ZIP archive of DHTML + Javascript in French and English (without the minimized JTMath library)
    SHA256 : 2561C0A513E0DA773AFCED3B780E15E5224D1A54DD862370C5A4EA3A0BB46CD0
  - ZIP archive of DHTML + Javascript in French and English (with the minimized JTMath library)
    SHA256 : 290F4316673CED8E99A6AF8582888EBDC6C8BAF289572DD29BC01387A1EFD6B6
  - Static HTML in French
    SHA256 : 820A515966D3A9B51B4A7E0071A01AA47ECF9F1D912A9FECAF639CF6A583DCB4
  - Static HTML in English
    SHA256 : AABF71A8EDF4BE7BDDB054A5367DA6496F55C08F3FFB48510A6D6A57302C026F
  - Static HTML in French and second part in English
    SHA256 : E1450175C4BC5FFA56527C494333E92DC77FFE3625B48FC0DB01AA8E7600AD30
  - Static HTML in English and second part in French
    SHA256 : 391985F2E1C9D5736B1466DDBFD90FE80FBB32BBB4C31939539FC39917D56749
  - PDF in French
    SHA256 : F726D065D2476FB1B67C158CBBF38FD8711EF4DF26B829F2010798D16E5088AD
  - PDF in English
    SHA256 : 3741FC5A0C9373233FB991C8B6C71EE3C5EDE55E6FDD758073A15E2AFEE7CA63
  - PDF in French and second part in English
    SHA256 : 8D02B7C994F8CC2DB0E170AE653F72303C3DC25A7BF30BA2E80982E01496DA4A

    - PDF in English and second part in French
      SHA256 : 7033199AEB33D9D26A4AB8B12531FB8D3462A64CE7F93229B4ABC11A4D0B0446
  - Version 3.1 : Proofs of theorems for the extensions of the Syracuse sequence :
    - ZIP archive of DHTML + Javascript in French and English (without the minimized JTMath library)
      SHA256 : 7EE4920F1D6323FBBC0F55FD34DC8583DAE2264173E31328D7FC42E41F5AB61A
    - ZIP archive of DHTML + Javascript in French and English (with the minimized JTMath library)
      SHA256 : 51FCE61F35C63041F53FAA24BBCFC9696D84EAF1FBFD70B4327FFCD7B2A842B8
    - **PDF of results** (appendice for the two following documents) on the website https://www.bajaxe.com :
      SHA256 : 4245BAA2E6D5CFBC4F6EEBA956013F59EC554AE3B057A948D1FA9A93065683AD
    - PDF in French
      SHA256 : 2809A0F11AE1534FA020209885232422EEB1706242C3FD0E9D682EE04B874306
    - PDF in English

Updated additional information on the website : https://www.bajaxe.com

--------------- Jacques BALLASI ---------------